Meeting on Geometric Analysis

Celebrating Barnabé Pessoa Lima's 60^{th} birthday

Lectures Notes on Geometric Analysis

Edited by

Leandro F. Pessoa

edufpi

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Meeting on Geometric Analysis

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Preface

This year the mathematics department of UFPI has the honor to celebrate the 60^{th} birthday of its dearest professor. Actually Professor Barnabé Pessoa Lima has been source of inspiration, mentor and teacher for most part of the mathematicians from Piauí State. Throughout all his professional life, through his serenity and extreme humbleness, Professor Barnabé has been a living proof that "simplicity is the ultimate sophistication".[†] The Meeting on Geometric Analysis was idealized to share this special occasion with all his friends, former and current students, as well as to demonstrate the tremendous respect and admiration felt for the honoree. These notes arises out of a singelous demonstration of friendship and respect of the authors to the honoree. Along with these notes, we give our best wishes to Barnabé, that he might have many more years to continue inspiring and encouraging us and other future mathematicians.

We wish to thank all the authors for their enthusiastic participation and their cooperation in writing up these notes. We would also like to thank the Universidade Federal do Piauí for supporting this special meeting.

Leandro F. Pessoa
December 2017

[†] This quotation is often attributed to Leonardo da Vinci

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Harmonic Forms and Theory of Submanifolds

Marcos Cavalcante[†], Abraão Mendes[†] and Feliciano Vitório[†]

Dedicated to Professor Barnabé Pessoa Lima on occasion of his 60th birthday

Abstract: In this note, we deal with compact free boundary hypersurfaces with respect to compact strictly convex domains in $\mathbb{R}^{n+1}$. Our main result says that under suitable geometric conditions the Betti numbers b_i vanish for $i = 1, \ldots, n-1$.

1 Introduction

1.1 Brief Historical and Statement of the Main Result

The study of minimal free boundary surfaces begins in the middle of twenty century with the works of Courant [5] and Lewy [9]. Along this period, very many breakthrough works were made by renowned authors, for instance, Nitsche [14], Ros [15] and Meeks and Yau [13]. More recently, the theme was strongly influenced by the works due to Fraser and Schoen [6, 7, 8]. The present paper is a concise version of a forthcoming paper by the authors that contains the following topological

[†]The authors were partially supported by CNPq-Brazil

Theorem 1.1. *[4] Let M^n be a compact free boundary hypersurface with respect to Ω. Suppose that $\partial\Omega$ is $(\underline{\kappa}, p)$-convex. There exists a positive constant $c = c(n, p, \underline{r}, \overline{r}, \underline{\kappa})$, such that if $\|\Phi\|^2 < c$, then, for $1 \leq p \leq n-1$, the cohomological group $H^p(M)$ is trivial.*

It is worthwhile to say that this work was partly inspired in the works [1], [2], [11].

1.2 Geometric Setting and Basic Definitions

Let Ω be a compact domain in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$. Let $\{\kappa_i\}_{1\leq i\leq n-1}$ be the principal curvatures of $\partial\Omega$, i.e., the eigenvalues of the shape operator $B = -\nabla\nu_{\partial\Omega}$ on $\partial\Omega$, where $\nu_{\partial\Omega}$ is the outward unit normal vector field. For each $x \in \partial\Omega$ and $p \in \{1 \leq i \leq n-1\}$ define

$$\kappa_{(p)}(x) = \inf_{1\leq i_1<\cdots<i_p\leq n-1} \left(\kappa_{i_1} + \cdots + \kappa_{i_p}\right)$$

We say that $\partial\Omega$ is $(\underline{\kappa}, p)$-*convex* if

$$\inf_{x\in\partial\Omega} \kappa_{(p)}(x) \geq \underline{\kappa} > 0.$$

Remark 1. *It is simple to verify that if $\partial\Omega$ is $(\underline{\kappa}, p)$-convex, then $\partial\Omega$ is $(\underline{\kappa}, q)$-convex, for any $q \geq p$.*

For a fixed point $x_0 \in \mathrm{int}(\Omega)$, we say that $(x_0, \underline{r}(x_0), \overline{r}(x_0))$ is an admissible triple if $B(x_0, \underline{r}(x_0)) \subset \Omega$, $B(x_0, \overline{r}(x_0)) \supset \Omega$ and for any $\varepsilon > 0$, $B(x_0, \overline{r}(x_0) - \varepsilon) \not\supset \Omega$. We call the positive number

$$\tau(\Omega) = \sup_{x_0\in\mathrm{int}(\Omega)} \tau(x_0) = \sup_{x_0\in\mathrm{int}(\Omega)} \left\{\frac{\underline{r}(x_0)}{\overline{r}(x_0)}\right\}$$

as the eccentricity of Ω. We notice that, by continuity and compactness classical results, there exist a point $\bar{x} \in \mathrm{int}(\Omega)$ such that $\tau(\bar{x}) = \tau(\Omega)$.

Let $x : M^n \to \bar{M}$ be an isometric immersion of an n-dimensional manifold M in a Riemannian manifold $\bar{M}$. Let us denote II the second fundamental form and $H = \frac{1}{n}\mathrm{tr}(II)$ the mean curvature vector field of the immersion x. The traceless second fundamental form Φ is defined by

$$\Phi(X,Y) = II(X,Y) - \langle X,Y\rangle H,$$

for all vector fields X,Y on M, where $\langle\,,\rangle$ is the metric of M. A simple computation shows that

$$|\Phi|^2 = |II|^2 - n|H|^2.$$

In particular, $|\Phi| \equiv 0$ if and only if the immersion x is totally umbilical.

We say that a compact hypersurface M is *free boundary* with respect to Ω if the following conditions hold:

i) $M \subset \Omega$ and $\mathrm{int(M)} \cap \partial\Omega = \varnothing$;

ii) $\partial M \subset \partial\Omega$ and ∂M meets $\partial\Omega$ orthogonally.

Remark 2. *It is an important fact that if M is free boundary with respect to Ω, then the shape operator on the boundary of M is the restriction of the shape operator of $\partial\Omega$ to ∂M. In particular, If $\partial\Omega$ is $(\underline{\kappa}, p)$-convex, then ∂M is $(\underline{\kappa}, p)$-convex.*

Let M^n be a compact Riemannian manifold with smooth boundary and let denote by $\Lambda^p(M)$ the space of differential p-forms on M. There exist two operators that play a fundamental role in the theory, namely, the exterior differential

$$d : \Lambda^p(M) \to \Lambda^{p+1}(M),$$

and the codifferential operator

$$d^* : \Lambda^p(M) \to \Lambda^{p-1}(M),$$

given, in terms of the Hodge star operator, by $d^* = (-1)^{n(p+1)+1} * d *$. The usual form Laplacian is given by

$$\Delta = dd^* + d^* d.$$

In order to obtain topological results via the Hodge-de Rham Theorem, we define two sets

$$\mathcal{H}_N^p(M) = \{\omega \in \Lambda^p(M); d\omega = 0, d^*\omega = 0 \text{ in } M \text{ and } \iota_\nu \omega = 0 \text{ on } \partial M\},$$

called as the set of the tangential harmonic p-forms and

$$\mathcal{H}_T^p(M) = \{\omega \in \Lambda^p(M); d\omega = 0, d^*\omega = 0 \text{ in } M \text{ and } \nu \wedge \omega = 0 \text{ on } \partial M\},$$

called as the set of the normal harmonic p-forms.

2 Analytical and Topological Ingredients

The fundamental tool in our result is the following Hodge-de Rham Theorem (See , for instance, [18]),

Theorem 2.1. *Let M be a compact orientable manifold with non-empty boundary. For every $p = 0, 1, \cdots, n$ the set of harmonic p-forms that are tangential to ∂M $\mathcal{H}_N^p(M)$ is isomorphic to the p-th cohomology group of M $H^p(M)$, in short,*

$$\mathcal{H}_N^p(M) \simeq H^p(M).$$

Moreover, since the Hodge star operator gives an isomorphism between $\mathcal{H}^p_N(M)$ and $\mathcal{H}^{n-p}_T(M)$, we have the following chain of isomorphisms

$$\mathcal{H}^p_T(M) \simeq \mathcal{H}^{n-p}_N(M) \simeq H^{n-p}(M) \simeq H_p(M,\partial M;\mathbb{R}),$$

where $H_p(M,\partial M;\mathbb{R})$ is the p-th relative homology group and the last isomorphism is a consequence of Poincaré-Lefschetz duality.

The proof uses the called Bochner Technique so we need the following Weitzenböck formula (for a proof see, for instance, the excellent book of P. Li [10])

Lemma 2.1. *Let $\omega \in \Lambda^p(M)$ be a p-form in M. Then*

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle\Delta\omega,\omega\rangle + \langle\mathcal{R}_p(\omega),\omega\rangle$$

where $\mathcal{R}_p$ is the Weitzenböck curvature.

Now, if $\omega \in \mathcal{H}^p_N(M)$ then the Weitzenböck formula says that

$$\int_M |\nabla\omega|^2 + \langle\mathcal{R}_p(\omega),\omega\rangle = -\int_{\partial M}\langle\mathcal{B}_p(\omega),\omega\rangle,$$

where

$$(\mathcal{B}_p(\omega))(e_1,\ldots,e_p) = \sum_{i=1}^{p}\omega(e_1,\ldots,Be_i,\ldots,e_p),$$

$B = -\nabla\nu_{\partial M}$ on ∂M is the shape operator of ∂M on M and $\nu_{\partial M}$ denotes the outward unit normal vector field,.

If $\{e_i\}_{i=1,\cdots,n-1}$ is a frame that diagonalizes the shape operator, then

$$(\mathcal{B}_p(\omega))(e_{i_1},\ldots,e_{i_p}) = \left(\sum_j \kappa_{i_j}\right)\omega(e_{i_1},\ldots,e_{i_p}),$$

Thus, if $\partial\Omega$ is $(\underline{\kappa}, p)$-convex, we have

$$\langle \mathcal{B}_p(\omega), \omega \rangle \geq \underline{\kappa}|\omega|^2.$$

Analogously, if $\omega \in \mathcal{H}_T^p(M)$ then

$$\int_M |\nabla \omega|^2 + \langle \mathcal{R}_p(\omega), \omega \rangle = -\int_{\partial M} \langle \mathcal{B}_p^*(\omega), \omega \rangle,$$

where $\mathcal{B}_p^* = *\mathcal{B}_{n-p}*$, that is,

$$\int_M |\nabla \omega|^2 + \langle \mathcal{R}_p(\omega), \omega \rangle = -\int_{\partial M} \langle \mathcal{B}_{n-p}(*\omega), *\omega \rangle.$$

Thus, if $\partial\Omega$ is $(\underline{\kappa}, n-p)$-convex, we have

$$\langle \mathcal{B}_p(\omega), \omega \rangle \geq \underline{\kappa}| * \omega|^2 = \underline{\kappa}|\omega|^2.$$

We can summarize the results above in the following

Lemma 2.2. *(Weitzenböck formula) Assume that* $\omega \in \mathcal{H}_N^p$ *, if* $\partial\Omega$ *is* $(\underline{\kappa}, p)$*-convex, then*

$$\int_M |\nabla \omega|^2 + \langle \mathcal{R}_p(\omega), \omega \rangle \geq -\underline{\kappa} \int_{\partial M} |\omega|^2. \tag{2.1}$$

Assume that $\omega \in \mathcal{H}_T^p$ *, if* $\partial\Omega$ *is* $(\underline{\kappa}, n-p)$*-convex, then*

$$\int_M |\nabla \omega|^2 + \langle \mathcal{R}_p(\omega), \omega \rangle \geq -\underline{\kappa} \int_{\partial M} |\omega|^2. \tag{2.2}$$

Lemma 2.3. *(Kato inequality) If* ω *is a harmonic* p*-form on* M^n *then*

$$|\nabla \omega|^2 \geq (1 + K_p)|\nabla|\omega||^2, \tag{2.3}$$

where

$$K_p = \begin{cases} \dfrac{1}{n-p}, & \textit{if } 1 \le p \le \left\lfloor \frac{n}{2} \right\rfloor, \\ \\ \dfrac{1}{p}, & \textit{if } \left\lfloor \frac{n}{2} \right\rfloor < p \le n-2. \end{cases}$$

The next two lemmas are addressed for submanifolds

Lemma 2.4. *(Lin) [12] If M^n is isometrically immersed in $\mathbb{R}^{n+1}$, then*

$$\begin{aligned} \langle \mathcal{R}_p(\omega), \omega \rangle \quad \ge \quad & \Big(p(n-p)H^2 - \tfrac{p(n-p)}{n} |\Phi|^2 \\ & - |n-2p| \sqrt{\tfrac{p(n-p)}{n}} |H| \cdot |\Phi| \Big) |\omega|^2. \end{aligned} \tag{2.4}$$

The last tool we need is the Hardy inequality for submanifolds recently discovered by Batista, Mirandola and the third named author in [3]. In our geometric setting it can be read as

Lemma 2.5. *(Batista, Mirandola, Vitório) [3] Let $X : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion, where M^n is a compact Riemannian n-manifold with boundary ∂M. We consider the point $\bar{x}$ as the origin of $\mathbb{R}^n$ and $r = |X|$, we have*

$$\int_M u^2 \le \left(\frac{2}{\bar{r}(n-2)} \right)^2 \int_M |\nabla u|^2 + \left(\frac{n}{\bar{r}(n-2)} \right)^2 \int_M |H|^2 u^2 + \frac{2}{n-2} \left(\frac{\bar{r}}{\underline{r}} \right)^2 \int_{\partial M} u^2,$$

for all nonnegative function $u \in C^1(M)$.

3 Proof of Theorem 1.1

Suppose that $\partial\Omega$ is $(\underline{\kappa}, p)$-convex and $\bar{r}(\bar{x}) = 1$. Let ω be a tangential harmonic p-form with $p \le \left\lfloor \frac{n}{2} \right\rfloor$ and let $u = |\omega|$. Then, plugging Kato's inequality in the Weitzenböck formula and using Lemma 2.4 we get

$$\begin{aligned} -\underline{\kappa}\textstyle\int_{\partial M} u^2 &\geq \tfrac{n-p+1}{n-p}\textstyle\int_M |\nabla u|^2 + p(n-p)\textstyle\int_M H^2u^2 - \tfrac{p(n-p)}{n}\textstyle\int_M |\Phi|^2u^2 \\ &\quad -(n-2p)\sqrt{\tfrac{p(n-p)}{n}}\textstyle\int_M |H||\Phi|u^2 \end{aligned}$$

Using Cauchy-Schwartz inequality and taking the supremum of $|\Phi|$ we have

$$\begin{aligned} -\underline{\kappa}\textstyle\int_{\partial M} u^2 &\geq \tfrac{n-p+1}{n-p}\textstyle\int_M |\nabla u|^2 + p(n-p)\textstyle\int_M H^2u^2 - \tfrac{p(n-p)}{n}\textstyle\int_M |\Phi|^2u^2 \\ &\quad -(n-2p)\sqrt{\tfrac{p(n-p)}{n}}\left(\tfrac{\varepsilon}{2}\textstyle\int_M |H|^2u^2 + \tfrac{1}{2\varepsilon}\textstyle\int_M |\Phi|^2u^2\right) \\ &\geq \tfrac{n-p+1}{n-p}\textstyle\int_M |\nabla u|^2 + A\textstyle\int_M H^2u^2 - B\|\Phi\|_\infty^2\textstyle\int_M u^2, \end{aligned}$$

where

$$A = A(n,p,\varepsilon) = p(n-p) - (n-2p)\sqrt{\frac{p(n-p)}{n}\frac{\varepsilon}{2}}$$

and

$$B = B(n,p.\varepsilon) = \frac{p(n-p)}{n} + (n-2p)\sqrt{\frac{p(n-p)}{n}\frac{1}{2\varepsilon}}.$$

Now we use Lemma 2.5 in the last integral.

$$C\int_M |\nabla u|^2 + D\int_M H^2u^2 + E\int_{\partial M} u^2 \leq 0,$$

where,

$$C = C(n,p,\|\Phi\|_\infty) = \tfrac{n-p+1}{n-p} - \tfrac{4}{(n-2)^2}B\|\Phi\|_\infty^2$$

$$D = D(n,p,\|\Phi\|_\infty) = A - \left(\tfrac{n}{n-2}\right)^2 B\|\Phi\|_\infty^2 \qquad \text{and}$$

$$E = E(n,p,\|\Phi\|_\infty) = \underline{\kappa} - \tfrac{2}{(n-2)\underline{r}^2}B\|\Phi\|_\infty^2.$$

It is clear that we can find a constant $c = c(n,p,\underline{r},\overline{r},\underline{\kappa})$, such that if $\|\Phi\|_\infty \leq c$, then $C,D \geq 0$ and $E > 0$. This implies that $u \equiv 0$ on ∂M, and by harmonicity $u \equiv 0$ in M. Therefore, by the Hodge-de Rham Theorem, we have that H^p is trivial.

Analogously, we can prove that $\mathcal{H}_T^p(M) = 0$. Using the duality induced by the Hodge star operator, we can extend the proof for $p \geq \lfloor \frac{n}{2} \rfloor$.

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Density and spectrum of minimal submanifolds in space forms

J. Fabio Montenegro† and Franciane B. Vieira

Dedicado ao professor Barnabé Pessoa Lima na ocasião dos seus 60 anos. Esta é a oportunidade ideal para agradecer ao professor Barnabé por tudo que faz pela matemática, pelos trabalhos de pesquisa que publicou, pelas gerações de matemáticos que ajudou a formar e, acima de tudo, pelo caráter bondoso, modesto e determinado que possui, que certamente o faz merecer todas essas homenagens.

Abstract: Let $\varphi : M^m \to N^n$ be a minimal, proper immersion in an ambient space suitably close to a space form $\mathbb{N}^n_k$ of curvature $-k \leq 0$. In this paper, we are interested in the relation between the density function $\Theta(r)$ of M and the spectrum of its Laplace-Beltrami operator. In particular, we prove that if $\Theta(r)$ has subexponential growth (when $k < 0$) or sub-polynomial growth ($k = 0$) along a sequence, then the spectrum of M^m is the same as that of the space form $\mathbb{N}^m_k$. Notably, the result applies to Anderson's (smooth) solutions of Plateau's problem at infinity on the hyperbolic space, independently of their boundary regularity. We also give a simple condition on the second fundamental form that ensures M to have finite density. In particular, we show that minimal submanifolds with finite total curvature in the hyperbolic space also have finite density.

†The author is partially supported by CNPq-Brazil

1 Introduction

Let M^m be a minimal, properly immersed submanifold in a complete ambient space N^n. In the present paper, we are interested in the case when N is close, in a sense made precise below, to a space form $\mathbb{N}^n_k$ of curvature $-k \le 0$. In particular, our focus is the study of the spectrum of the Laplace Beltrami operator $-\Delta$ on M and its relationship with the density at infinity of M, that is, the limit as $r \to +\infty$ of the (monotone) quantity

$$\Theta(r) \doteq \frac{\mathrm{vol}(M \cap B_r)}{V_k(r)}, \tag{1.1}$$

where B_r indicates a geodesic ball of radius r in N^n and $V_k(r)$ is the volume of a geodesic ball of radius r in $\mathbb{N}^m_k$. Hereafter, we will say that M has finite density if

$$\Theta(+\infty) \doteq \lim_{r \to +\infty} \Theta(r) < +\infty.$$

To properly put our results into perspective, we briefly recall few facts about the spectrum of the Laplacian on a geodesically complete manifold. It is known by works of P. Chernoff [15] and R.S. Strichartz [49] that $-\Delta$ on a complete manifold is essentially self-adjoint on the domain $C^\infty_c(M)$, and thus it admits a unique self-adjoint extension, which we still call $-\Delta$. Since $-\Delta$ is positive and self-adjoint, its spectrum is the set of $\lambda \ge 0$ such that $\Delta + \lambda I$ does not have bounded inverse. Sometimes we say spectrum of M rather than spectrum of $-\Delta$ and we denote it by $\sigma(M)$. The well-known Weyl's characterization for the spectrum of a self-adjoint operator in a Hilbert space implies the following

Lemma 1.1. *[19, Lemma 4.1.2] A number $\lambda \in \mathbb{R}$ lies in $\sigma(M)$ if and only if there exists a sequence of nonzero functions $u_j \in$* $\mathrm{Dom}(-\Delta)$ *such that*

$$\|\Delta u_j + \lambda u_j\|_2 = o\big(\|u_j\|_2\big) \qquad as\ j \to +\infty. \tag{1.2}$$

In the literature, characterizations of the whole $\sigma(M)$ are known only in few special cases. Among them, the Euclidean space, for which $\sigma(\mathbb{R}^m) = [0, \infty)$, and the hyperbolic space $\mathbb{H}^m_k$, for which

$$\sigma(\mathbb{H}^m_k) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.3}$$

The approach to guarantee that $\sigma(M) = [c, +\infty)$, for some $c \geq 0$, usually splits into two parts. The first one is to show that $\inf \sigma(M) \geq c$ via, for instance, the Laplacian comparison theorem from below ([42], [5]), and the second one is to produce a sequence like in lemma 1.1 for each $\lambda > c$. This step is accomplished by considering radial functions of compact support, and, at least in the first results on the topic like the one in [21], uses the comparison theorems on both sides for $\Delta\rho$, ρ being the distance from a fixed origin $o \in M$. Therefore, the method needs both a pinching on the sectional curvature and the smoothness of ρ, that is, that o is a pole of M (see [21], [25],[36] and Corollary 2.17 in [8]), which is a severe topological restriction.

The main concern in this paper is to achieve, in the above-mentioned setting of minimal submanifolds $\varphi : M \to N$, a characterization of the whole $\sigma(M)$ free from curvature or topological conditions on M (in this respect, observe that the completeness of M follows from that of N and the properness of φ). It is known by [18] and [5] that for a minimal immersion $\varphi : M^m \to \mathbb{N}^n_k$ the fundamental tone of M, $\inf \sigma(M)$, is at least that of $\mathbb{N}_k^m$, i.e.,

$$\inf \sigma(M) \geq \frac{(m-1)^2 k}{4}. \tag{1.4}$$

Moreover, as a corollary of [34] and [4], [6], if the second fundamental form II satisfies the decay estimate

$$\begin{aligned} &\lim_{\rho(x)\to+\infty} \rho(x)|II(x)| = 0 \qquad &&\text{if } k=0 \\ &\lim_{\rho(x)\to+\infty} |II(x)| = 0 \qquad &&\text{if } k>0 \end{aligned} \tag{1.5}$$

($\rho(x)$ being the intrinsic distance with respect to some fixed origin $o \in M$), then M has the same spectrum that a totally geodesic submanifold $\mathbb{N}_k^m \subset \mathbb{N}_k^n$, that is,

$$\sigma(M) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.6}$$

According to [1], [20], (1.5) is ensured when M has finite total curvature, that is, when

$$\int_M |II|^m < +\infty. \tag{1.7}$$

Condition (1.5) is a quite binding requirement for (1.6) to hold, since it needs a pointwise control of the second fundamental form, and the search for more manageable conditions has been at the heart of the present paper. Here, we identify a suitable growth on the density function $\Theta(r)$ *along a sequence* as a natural candidate to replace it, see (1.9). As a very special case, (1.6) holds when M has finite density. It might be interesting that just a volume growth condition along a sequence could control the whole spectrum of M; for this to happen, the minimality condition enters in a crucial and subtle way.

Regarding the relation between (1.7) and the finiteness of $\Theta(+\infty)$, we remark that their interplay has been investigated in depth for minimal submanifolds of $\mathbb{R}^n$, but the case of $\mathbb{H}^n_k$ seems to be partly unexplored. In the next section, we will briefly discuss the state of the art, to the best of our knowledge. As a corollary of Theorem 1.2 below, we will show the following

Corollary 1.1. *Let M^m be a minimal properly immersed submanifold in $\mathbb{H}^n_k$. If M has finite total curvature, then $\Theta(+\infty) < +\infty$.*

As far as we know, this result was previously known just in dimension $m = 2$ via a Chern-Osserman type inequality, see the next section for further details.

We now come to our results, beginning with defining the ambient spaces which we are interested in: these are manifolds with a pole, whose radial sectional curvature is suitably pinched to that of the model $\mathbb{N}^n_k$.

Definition 1.1. *Let N^n possess a pole $\bar{o}$ and denote with $\bar{\rho}$ the distance function from $\bar{o}$. Assume that the radial sectional curvature $\bar{K}_{\mathrm{rad}}$ of N, that is, the sectional curvature restricted to planes π containing $\bar{\nabla}\bar{\rho}$, satisfies*

$$-G\big(\bar{\rho}(x)\big) \le \bar{K}_{\mathrm{rad}}(\pi_x) \le -k \le 0 \qquad \forall\, x \in N\backslash\{\bar{o}\}, \tag{1.8}$$

for some $G \in C^0(\mathbb{R}^+_0)$. We say that

(i) N has a pointwise (respectively, integral) pinching to $\mathbb{R}^n$ if $k = 0$ and

$$sG(s) \to 0 \ \text{ as } s \to +\infty \qquad \big(\text{respectively, } sG(s) \in L^1(+\infty)\big);$$

(*ii*) N has a pointwise (respectively, integral) pinching to $\mathbb{H}^n_k$ if $k > 0$ and

$$G(s) - k \to 0 \text{ as } s \to +\infty \qquad \big(\text{respectively, } G(s) - k \in L^1(+\infty)\big).$$

Hereafter, given an ambient manifold N with a pole $\bar{o}$, the density function $\Theta(r)$ will always be computed by taking extrinsic balls centered at $\bar{o}$.

Our main achievements are the following two theorems. The first one characterizes $\sigma(M)$ when the density of M grows subexponentially (respectively, sub-polynomially) along a sequence. Condition (1.9) below is very much in the spirit of a classical growth requirement due to R. Brooks [9] and Y. Higuchi [30] to bound from above the infimum of the essential spectrum of $-\Delta$. However, we stress that our Theorem 1.1 seems to be the first result in the literature characterizing the whole spectrum of M under just a mild volume assumption.

Theorem 1.1. *Let $\varphi : M^m \to N^n$ be a minimal properly immersed submanifold, and suppose that N has a pointwise or an integral pinching to a space form. If either*

$$\begin{aligned} &N \text{ is pinched to } \mathbb{H}^n_k, \text{ and} && \liminf_{s\to+\infty} \frac{\log\Theta(s)}{s} = 0, \quad \text{or} \\ &N \text{ is pinched to } \mathbb{R}^n, \text{ and} && \liminf_{s\to+\infty} \frac{\log\Theta(s)}{\log s} = 0. \end{aligned} \tag{1.9}$$

then

$$\sigma(M) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.10}$$

The above theorem is well suited for minimal submanifolds constructed via Geometric Measure Theory since, typically, their existence is guaranteed by controlling the density function $\Theta(r)$. As an important example, Theorem 1.1 applies to all solutions of Plateau's problem at infinity $M^m \to \mathbb{H}^n_k$ constructed in [2], provided that they are smooth. Indeed, because of their construction, $\Theta(+\infty) < +\infty$ (see [2], part [A] at p. 485) and they are proper (it can also be deduced as a consequence of $\Theta(+\infty) < +\infty$). By standard regularity theory, smoothness of M^m is automatic if $m \leq 6$.

Corollary 1.2. *Let $\Sigma \subset \partial_\infty \mathbb{H}^n_k$ be a closed, integral $(m-1)$ current in the boundary at infinity of $\mathbb{H}^n_k$ such that, for some neighbourhood $U \subset \mathbb{H}^n_k$ of* $\mathrm{supp}(\Sigma)$*, Σ does not bound in U, and let $M^m \hookrightarrow \mathbb{H}^n_k$ be the solution of Plateau's problem at infinity constructed in [2] for Σ. If M is smooth, then* (1.10) *holds.*

An interesting fact of Corollary 1.2 is that M is *not* required to be regular up to $\partial_\infty \mathbb{H}^n_k$, in particular it might have infinite total curvature. In this respect, we observe that if M be C^2 up to $\partial_\infty \mathbb{H}^n_k$, then M would have finite total curvature (Lemma 5 in Appendix 1 [38]). By deep regularity results, this is the case if, for instance, $M^m \to \mathbb{H}^{m+1}_k$ is a smooth hypersurface that solves Plateau's problem for Σ, and Σ is a $C^{2,\alpha}$ (for $\alpha > 0$), embedded compact hypersurface of $\partial_\infty \mathbb{H}^n_k$. See Appendix 1 for details.

The spectrum of solutions of Plateau's problems has also been considered in [3] for minimal surfaces in $\mathbb{R}^3$. In this respect, it is interesting to compare Corollary 1.2 with (3) of Corollary 2.6 therein.

In our second result we focus on the particular case when $\Theta(+\infty) < +\infty$, and we give a sufficient condition for its validity in terms of the decay of the second fundamental form. Towards this aim, we shall restrict to ambient spaces with an integral pinching.

Theorem 1.2. *Let $\varphi : M^m \to N^n$ be a minimal immersion, and suppose that N has an integral pinching to a space form. Denote with $\rho(x)$ the intrinsic distance from some reference origin $o \in M$. Assume that there exist $c > 0$ and $\alpha > 1$ such that the second fundamental form satisfies, for $\rho(x) >> 1$,*

$$\begin{aligned} |II(x)|^2 &\le \frac{c}{\rho(x)\log^\alpha \rho(x)} && \text{if } N \text{ is pinched to } \mathbb{H}^n_k; \\ |II(x)|^2 &\le \frac{c}{\rho(x)^2\log^\alpha \rho(x)} && \text{if } N \text{ is pinched to } \mathbb{R}^n. \end{aligned} \tag{1.11}$$

Then, φ is proper, M is diffeomorphic to the interior of a compact manifold with boundary, and $\Theta(+\infty) < +\infty$.

2 Preliminaries

Let $\varphi : (M^m, \langle\, , \rangle) \to (N^n, (\, ,))$ be an isometric immersion of a complete m-dimensional Riemannian manifold M into an ambient manifold N of dimension n and possessing a pole $\bar{o}$. We denote with $\nabla, \mathrm{Hess}, \Delta$ the connection, the Riemannian Hessian and the Laplace-Beltrami operator on M, while quantities related to N will be marked with a bar. For instance, $\bar{\nabla}, \overline{\mathrm{dist}}, \overline{\mathrm{Hess}}$ will identify the connection, the distance function and the Hessian in N. Let $\bar{\rho}(x) = \overline{\mathrm{dist}}(x, \bar{o})$ be the distance function from $\bar{o}$. Geodesic balls in N of radius R and center y will be denoted with $B^N_R(y)$. Moreover, set $r : M \to \mathbb{R}$, $r(x) = \bar{\rho}\big(\varphi(x)\big)$, for the extrinsic distance from $\bar{o}$. We will indicate with Γ_s the extrinsic geodesic spheres restricted to M: $\Gamma_s \doteq \{x \in M;\ r(x) = s\}$. Fix a base point $o \in M$. In what follows, we shall also consider the intrinsic distance function $\rho(x) = \mathrm{dist}(x, o)$ from a reference origin $o \in M$.

2.1 Target spaces

Hereafter, we consider an ambient space N possessing a pole $\bar{o}$ and, setting $\bar{\rho}(x) \doteq \mathrm{dist}(x, \bar{o})$, we assume that (1.8) is met for some $k \geq 0$ and some $G \in C^0(\mathbb{R}^+_0)$. Let $\mathrm{sn}_k(t)$ be the solution of

$$\begin{cases} \mathrm{sn}_k'' - k\,\mathrm{sn}_k = 0 \quad \text{on } \mathbb{R}^+, \\ \mathrm{sn}_k(0) = 0, \quad \mathrm{sn}_k'(0) = 1, \end{cases} \tag{2.1}$$

that is

$$\mathrm{sn}_k(t) = \begin{cases} t & \text{if } k = 0, \\ \sinh(\sqrt{k}t)/\sqrt{k} & \text{if } k > 0. \end{cases} \tag{2.2}$$

Observe that $\mathbb{R}^n$ and $\mathbb{H}^n_k$ can be written as the differentiable manifold $\mathbb{R}^n$ equipped with the metric given, in polar geodesic coordinates $(\rho, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ centered at some origin, by

$$\mathrm{d}s_k^2 = \mathrm{d}\rho^2 + \mathrm{sn}_k^2(\rho)\,\mathrm{d}\theta^2,$$

$\mathrm{d}\theta^2$ being the metric on the unit sphere $\mathbb{S}^{n-1}$.
We also consider the model M^n_g associated with the lower bound $-G$ for $\bar{K}_{\mathrm{rad}}$, that is, we let $g \in C^2(\mathbb{R}^+_0)$ be the solution of

$$\begin{cases} g'' - Gg = 0 \quad \text{on } \mathbb{R}^+, \\ g(0) = 0, \quad g'(0) = 1, \end{cases} \tag{2.3}$$

and we define M^n_g as being $(\mathbb{R}^n, \mathrm{d}s_g^2)$ with the C^2-metric $\mathrm{d}s_g^2 = \mathrm{d}\rho^2 + g^2(\rho)\mathrm{d}\theta^2$ in polar coordinates. Condition (1.8) and the Hessian comparison theorem (Theorem 2.3 in [45], or Theorem 1.15 in [8]) imply

$$\frac{\mathrm{sn}_k'(\bar{\rho})}{\mathrm{sn}_k(\bar{\rho})}\Big((\, , \,) - \mathrm{d}\bar{\rho} \otimes \mathrm{d}\bar{\rho}\Big) \leq \overline{\mathrm{Hess}}(\bar{\rho}) \leq \frac{g'(\bar{\rho})}{g(\bar{\rho})}\Big((\, , \,) - \mathrm{d}\bar{\rho} \otimes \mathrm{d}\bar{\rho}\Big). \tag{2.4}$$

The next proposition investigates the ODE properties that follow from the assumptions of pointwise or integral pinching.

Proposition 2.1. *Let N^n satisfy* (1.8)*, and let* sn_k, g *be solutions of* (2.2), (2.3)*. Define*

$$\zeta(s) \doteq \frac{g'(s)}{g(s)} - \frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}. \tag{2.5}$$

Then, $\zeta(0^+) = 0$*,* $\zeta \geq 0$ *on* $\mathbb{R}^+$*. Moreover,*

(*i*) *If N has a pointwise pinching to $\mathbb{H}^n_k$ or $\mathbb{R}^n$, then $\zeta(s) \to 0$ as $s \to +\infty$.*

(*ii*) *If N has an integral pinching to $\mathbb{H}^n_k$ or $\mathbb{R}^n$, then $g/\mathrm{sn}_k \to C$ as $s \to +\infty$ for some $C \in \mathbb{R}^+$, and*

$$\zeta(s) \in L^1(\mathbb{R}^+), \qquad \zeta(s)\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} \to 0 \ \text{ as } s \to +\infty. \tag{2.6}$$

2.2 A transversality lemma

This subsection is devoted to an estimate of the measure of the critical set

$$S_{t,s} = \Big\{ x \in M \ : \ t \leq r(x) \leq s, \ |\nabla r(x)| = 0 \Big\},$$

with the purpose of justifying some coarea's formulas for integrals over extrinsic annuli. We begin with the next

Lemma 2.1. *Let $\varphi : M^m \to N^n$ be an isometric immersion, and let $r(x) = \overline{\mathrm{dist}}(\varphi(x), \bar{o})$ be the extrinsic distance function from $\bar{o} \in N$. Denote with $\Gamma_\sigma \doteq \{x \in M; \, r(x) = \sigma\}$. Then, for each $f \in L^1(\{t \leq r \leq s\})$,*

$$\int_{\{t\le r\le s\}} f\,\mathrm{d}x = \int_{S_{t,s}} f\,\mathrm{d}x + \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|}\right]\mathrm{d}\sigma. \tag{2.7}$$

In particular, if

$$\mathrm{vol}(S_{t,s}) = 0, \tag{2.8}$$

then

$$\int_{\{t\le r\le s\}} f\,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|}\right]\mathrm{d}\sigma. \tag{2.9}$$

Let now N possess a pole $\bar{o}$ and satisfy (1.8), and consider a minimal immersion $\varphi : M \to N$. Since, by the Hessian comparison theorem, geodesic spheres in N centered at $\bar{o}$ are positively curved, it is reasonable to expect that the “transversality" condition (2.8) holds. This is the content of the next

Proposition 2.2. *Let $\varphi : M^m \to N^n$ be a minimal immersion, where N possesses a pole $\bar{o}$ and satisfies* (1.8). *Then,*

$$\mathrm{vol}(S_{0,+\infty}) = 0. \tag{2.10}$$

3 Monotonicity formulae and conditions equivalent to $\Theta(+\infty) < +\infty$

Our first step is to improve the classical monotonicity formula for $\Theta(r)$, that can be found in [48] (for $N = \mathbb{R}^n$) and [2] (for $N = \mathbb{H}^n_k$). For $k \ge 0$, let v_k, V_k denote the volume function, respectively, of geodesic spheres and balls in the space form of sectional curvature $-k$ and dimension m, i.e.,

$$v_k(s) = \omega_{m-1}\mathrm{sn}_k(s)^{m-1}, \qquad V_k(s) = \int_0^s v_k(\sigma)\mathrm{d}\sigma, \tag{3.1}$$

where ω_{m-1} is the volume of the unit sphere $\mathbb{S}^{m-1}$. Although we shall not use all the four monotone quantities in (3.3) below, nevertheless they have independent interest, and for this reason we state the result in its full strength. We define the *flux* $J(s)$ of ∇r over the extrinsic sphere Γ_s:

$$J(s) \doteq \frac{1}{v_k(s)} \int_{\Gamma_s} |\nabla r|. \tag{3.2}$$

Proposition 3.1 (The monotonicity formulae)**.** *Suppose that N has a pole $\bar{o}$ and satisfies* (1.8)*, and let $\varphi : M^m \to N^n$ be a proper minimal immersion. Then, the functions*

$$\Theta(s), \qquad \frac{1}{V_k(s)} \int_{\{0\le r\le s\}} |\nabla r|^2 \tag{3.3}$$

are absolutely continuous and monotone non-decreasing. Moreover, $J(s)$ coincides, on an open set of full measure, with the absolutely continuous function

$$\bar{J}(s) \doteq \frac{1}{v_k(s)} \int_{\{r\le s\}} \Delta r$$

and $\bar{J}(s)$, $V_k(s)\big[\bar{J}(s) - \Theta(s)\big]$ are non-decreasing. In particular, $J(s) \ge \Theta(s)$ a.e. on $\mathbb{R}^+$.

Proof. We first observe that, in view of Lemma 2.1 and Proposition 2.2 applied with $f = \Delta r$,

$$v_k(s)\bar{J}(s) \doteq \int_{\{r\le s\}} \Delta r \equiv \int_0^s \left[\int_{\Gamma_\sigma} \frac{\Delta r}{|\nabla r|} \right] \mathrm{d}\sigma \tag{3.4}$$

is absolutely continuous, and by the divergence theorem it coincides with $v_k(s)J(s)$ whenever s is a regular value of r. Consider

$$f(s) = \int_0^s \frac{V_k(\sigma)}{v_k(\sigma)} \mathrm{d}\sigma = \int_0^s \frac{1}{v_k(\sigma)} \left[\int_0^\sigma v_k(\tau)\mathrm{d}\tau \right] \mathrm{d}\sigma \tag{3.5}$$

which is a C^2 solution of

$$f'' + (m-1)\frac{\mathrm{sn}_k'}{\mathrm{sn}_k} f' = 1 \quad \text{on } \mathbb{R}^+, \quad f(0)=0, \quad f'(0)=0,$$

and define $\psi(x) = f(r(x)) \in C^2(M)$. Let $\{e_i\}$ be a local orthonormal frame on M. Since φ is minimal, by the chain rule and the lower bound in the Hessian comparison theorem 2.4

$$\Delta r = \sum_{j=1}^{m} \overline{\mathrm{Hess}}(\bar{\rho})\big(\mathrm{d}\varphi(e_j), \mathrm{d}\varphi(e_j)\big) \ge \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\big(m - |\nabla r|^2\big). \tag{3.6}$$

We then compute

$$\begin{aligned} \Delta\psi &= f''|\nabla r|^2 + f'\Delta r \ge f''|\nabla r|^2 + f'\frac{\mathrm{sn}_k'}{\mathrm{sn}_k}(m - |\nabla r|^2) \\ &= 1 + \big(1 - |\nabla r|^2\big)\left(f'(r)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} - f''(r)\right). \end{aligned} \tag{3.7}$$

It is not hard to show that the function

$$z(s) \doteq f'(s)\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} - f''(s) = \frac{m}{m-1}\frac{V_k(s)v_k'(s)}{v_k^2(s)} - 1.$$

is non-negative and non-decreasing on $\mathbb{R}^+$. Indeed, from

$$z(0) = 0, \qquad z'(s) = \frac{m}{v_k(s)}\left[kV_k(s) - \frac{1}{m-1}v_k'(s)z(s)\right] \tag{3.8}$$

we deduce that $z' > 0$ when $z < 0$, which proves that $z \ge 0$ on $\mathbb{R}^+$. Fix $0 < t < s$ regular values for r. Integrating (3.7) on the smooth compact set $\{t \le r \le s\}$ and using the divergence theorem we deduce

$$\frac{V_k(s)}{v_k(s)}\int_{\Gamma_s}|\nabla r| - \frac{V_k(t)}{v_k(t)}\int_{\Gamma_t}|\nabla r| \ge \mathrm{vol}\big(\{t \le r \le s\}\big). \tag{3.9}$$

By the definition of $J(s)$ and $\Theta(s)$, and since $J(s) \equiv \bar{J}(s)$ for regular values, the above inequality rewrites as follows:

$$V_k(s)\bar{J}(s) - V_k(t)\bar{J}(t) \geq V_k(s)\Theta(s) - V_k(t)\Theta(t),$$

or in other words,

$$V_k(s)\big[\bar{J}(s) - \Theta(s)\big] \geq V_k(t)\big[\bar{J}(t) - \Theta(t)\big].$$

Since all the quantities involved are continuous, the above relation extends to all $t, s \in \mathbb{R}^+$, which proves the monotonicity of $V_k[\bar{J} - \Theta]$. Letting $t \to 0$ we then deduce that $\bar{J}(s) \geq \Theta(s)$ on $\mathbb{R}^+$. Next, by using $f \equiv 1$ and $f \equiv |\nabla r|^2$ in Lemma 2.1 and exploiting again Proposition 2.2 we get

$$\mathrm{vol}\big(\{t \leq r \leq s\}\big) = \int_t^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma, \quad \int_{\{0 \leq r \leq s\}} |\nabla r|^2 = \int_0^s \left[\int_{\Gamma_\sigma} |\nabla r|\right] \mathrm{d}\sigma, \tag{3.10}$$

showing that the two quantities in (3.3) are absolutely continuous. Plugging into (3.9), letting $t \to 0$ and using that $z \geq 0$ we deduce

$$\frac{V_k(s)}{v_k(s)} \int_{\Gamma_s} |\nabla r| \geq \int_0^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma, \tag{3.11}$$

for regular s, which together with the trivial inequality $|\nabla r|^{-1} \geq |\nabla r|$ and with (3.10) gives

$$\begin{aligned} V_k(s) \int_{\Gamma_s} |\nabla r| &\geq v_k(s) \int_0^s \left[\int_{\Gamma_\sigma} |\nabla r|\right] \mathrm{d}\sigma, \\ V_k(s)\left[\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{vol}\big(\{r \leq s\}\big)\right] &\geq v_k(s)\mathrm{vol}\big(\{r \leq s\}\big). \end{aligned} \tag{3.12}$$

Integrating the second inequality we obtain the monotonicity of $\Theta(s)$, while integrating the first one and using (3.10) we obtain the monotonicity of the second quantity in (3.3). To show the monotonicity of $\bar{J}(s)$, by (3.6) and using the full information coming from (2.4) we obtain

$$\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\big(m-|\nabla r|^2\big) \le \Delta r \le \frac{g'(r)}{g(r)}\big(m-|\nabla r|^2\big). \tag{3.13}$$

In view of the identity (3.4), we consider regular $s>0$, we divide (3.13) by $|\nabla r|$ and integrate on Γ_s to get

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\int_{\Gamma_s}\frac{m-|\nabla r|^2}{|\nabla r|} \le \big(v_k(s)\bar{J}(s)\big)' \le \frac{g'(s)}{g(s)}\int_{\Gamma_s}\frac{m-|\nabla r|^2}{|\nabla r|}. \tag{3.14}$$

Writing $m-|\nabla r|^2 = m(1-|\nabla r|^2)+(m-1)|\nabla r|^2$, and setting for convenience

$$v_g(s) = \omega_{m-1}g(s)^{m-1}, \qquad T(s) \doteq \frac{\int_{\Gamma_s}|\nabla r|^{-1}}{\int_{\Gamma_s}|\nabla r|} - 1, \tag{3.15}$$

rearranging we deduce the two inequalities

$$\begin{aligned}
\big(v_k(s)\bar{J}(s)\big)' &\ge v_k'(s)\bar{J}(s) + m\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}T(s)v_k(s)\bar{J}(s) \\
\big(v_k(s)\bar{J}(s)\big)' &\le \frac{v_g'(s)}{v_g(s)}v_k(s)\bar{J}(s) + m\frac{g'(s)}{g(s)}T(s)v_k(s)\bar{J}(s).
\end{aligned} \tag{3.16}$$

Expanding the derivative on the left-hand side, we deduce

$$\begin{aligned}
\bar{J}'(s) &\ge m\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}T(s)\bar{J}(s), \\
\left(\frac{v_k(s)}{v_g(s)}\bar{J}(s)\right)' &\le m\frac{g'(s)}{g(s)}T(s)\left(\frac{v_k(s)}{v_g(s)}\bar{J}(s)\right).
\end{aligned} \tag{3.17}$$

The first inequality together with the non-negativity of T implies the desired $\bar{J}' \geq 0$, concluding the proof. The second inequality in (3.17), on the other hand, will be useful in awhile. □

We next investigate conditions equivalent to the finiteness of the density.

Proposition 3.2. *Suppose that* N *has a pole and satisfies* (1.8). *Let* $\varphi : M^m \to N^n$ *be a proper minimal immersion. Then, the following properties are equivalent:*

(1) $\Theta(+\infty) < +\infty$;

(2) $\bar{J}(+\infty) < +\infty$.

Moreover, both (1) *and* (2) *imply that*

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\left[\frac{\int_{\Gamma_s}|\nabla r|^{-1}}{\int_{\Gamma_s}|\nabla r|}-1\right] \in L^1(\mathbb{R}^+). \tag{3}$$

If further N *has an integral pinching to* $\mathbb{R}^n$ *or* $\mathbb{H}^n_k$, *then* $(1) \Leftrightarrow (2) \Leftrightarrow (3)$.

4 Proof of Theorem 1

Let M^m be a minimal properly immersed submanifold in N^n, and suppose that N has a pointwise or integral pinching to a space form. Because of the upper bound in (1.8), by [18] and [5] the bottom of $\sigma(M)$ satisfies

$$\inf \sigma(M) \geq \frac{(m-1)^2 k}{4}. \tag{4.1}$$

Briefly, the lower bound in (3.13) implies

$$\Delta r \geq (m-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \geq (m-1)\sqrt{k} \qquad \text{on } M.$$

Integrating on a relatively compact, smooth open set Ω and using the divergence theorem and $|\nabla r| \le 1$, we deduce $\mathcal{H}^{m-1}(\partial\Omega) \ge (m-1)\sqrt{k}\mathrm{vol}(\Omega)$. The desired (4.1) then follows from Cheeger's inequality:

$$\inf \sigma(M) \ge \frac{1}{4}\left(\inf_{\Omega \Subset M} \frac{\mathcal{H}^{m-1}(\partial\Omega)}{\mathrm{vol}(\Omega)}\right)^2 \ge \frac{(m-1)^2 k}{4}.$$

To complete the proof of the theorem, since $\sigma(M)$ is closed it is sufficient to show that each $\lambda > (m-1)^2 k/4$ lies in $\sigma(M)$.

Set for convenience $\beta \doteq \sqrt{\lambda - (m-1)^2 k/4}$ and, for $0 \le t < s$, let $A_{t,s}$ denote the extrinsic annulus

$$A_{t,s} \doteq \left\{ x \in M \ : \ r(x) \in [t,s] \right\}.$$

Define the weighted measure $\mathrm{d}\mu_k \doteq v_k(r)^{-1}\mathrm{d}x$ on $\{r \ge 1\}$. Hereafter, we will always restrict to this set. Consider

$$\psi(s) \doteq \frac{e^{i\beta s}}{\sqrt{v_k(s)}}, \qquad \text{which solves} \qquad \psi'' + \psi' \frac{v_k'}{v_k} + \lambda\psi = a(s)\psi, \tag{4.2}$$

where

$$a(s) \doteq \frac{(m-1)^2 k}{4} + \frac{1}{4}\left(\frac{v_k'(s)}{v_k(s)}\right)^2 - \frac{1}{2}\frac{v_k''(s)}{v_k(s)} \to 0 \tag{4.3}$$

as $s \to +\infty$. For technical reasons, fix $R > 1$ large such that $\Theta(R) > 0$. Fix t, s, S such that

$$R + 1 < t < s < S - 1,$$

and let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 0 \ \text{ outside of } (t-1, S), \quad \eta \equiv 1 \ \text{ on } (t,s),$$

$$|\eta'|+|\eta''|\le C_0 \quad \text{on } [t-1,s], \qquad |\eta'|+|\eta''|\le \tfrac{C_0}{S-s} \quad \text{on } [s,S]$$

for some absolute constant C_0 (the last relation is possible since $S-s\ge 1$). The value S will be chosen later in dependence of s. Set $u_{t,s}\doteq \eta(r)\psi(r)\in C_c^\infty(M)$. Then, by (4.2),

$$\begin{aligned}
\Delta u_{t,s}+\lambda u_{t,s} &= (\eta''\psi+2\eta'\psi'+\eta\psi'')|\nabla r|^2+(\eta'\psi+\eta\psi')\Delta r+\lambda\eta\psi\\
&= \left(\eta''\psi+2\eta'\psi'-\frac{v_k'}{v_k}\eta\psi'-\lambda\eta\psi+a\eta\psi\right)(|\nabla r|^2-1)+a\eta\psi\\
&\quad +(\eta'\psi+\eta\psi')\left(\Delta r-\frac{v_k'}{v_k}\right)+\left(\eta''\psi+2\eta'\psi'+\eta'\psi\frac{v_k'}{v_k}\right).
\end{aligned}$$

Using that there exists an absolute constant c for which $|\psi|+|\psi'|\le c/\sqrt{v_k}$, the following inequality holds:

$$\begin{aligned}
\|\Delta u_{t,s}+\lambda u_{t,s}\|_2^2 \;\le\; & C\left(\int_{A_{t-1,S}}\left[(1-|\nabla r|^2)^2+\left(\Delta r-\frac{v_k'}{v_k}\right)^2+a(r)^2\right]\mathrm{d}\mu_k\right.\\
&\left.+\frac{\mu_k(A_{s,S})}{(S-s)^2}+\mu_k(A_{t-1,t})\right),
\end{aligned}$$

for some suitable C depending on c, C_0. Since $\|u_{t,s}\|_2^2\ge \mu_k(A_{t,s})$ and $(1-|\nabla r|^2)^2\le 1-|\nabla r|^2$, we obtain

$$\begin{aligned}
\frac{\|\Delta u_{t,s}+\lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} \le & \left(\frac{C}{\mu_k(A_{t,s})}\int_{A_{t-1,S}}\left[1-|\nabla r|^2+\left(\Delta r-\frac{v_k'}{v_k}\right)^2+a(r)^2\right]\mathrm{d}\mu_k\right.\\
&\left.+\frac{1}{(S-s)^2}\frac{\mu_k(A_{s,S})}{\mu_k(A_{t,s})}+\frac{\mu_k(A_{t-1,t})}{\mu_k(A_{t,s})}\right) \qquad (4.4)
\end{aligned}$$

Next, using (2.4),

$$\Delta r = \sum_{j=1}^{m} \overline{\mathrm{Hess}}(\bar\rho)(e_i,e_i) = \frac{v_k'(r)}{v_k(r)} + \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + \mathcal{P}(x),$$

where, by Proposition 2.1,

$$\begin{aligned} 0 \le \mathcal{P}(x) &\doteq \sum_{j=1}^{m} \overline{\mathrm{Hess}}(\bar\rho)(e_i,e_i) - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(m-|\nabla r|^2) \\ &\le \left(\frac{g'(r)}{g(r)} - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\right)(m-|\nabla r|^2) \\ &= \zeta(r)(m-|\nabla r|^2) \le m\zeta(r). \end{aligned} \tag{4.5}$$

We thus obtain, on the set $\{r \ge 1\}$,

$$\begin{aligned} \left(\Delta r - \frac{v_k'}{v_k}\right)^2 + 1 - |\nabla r|^2 + a(r)^2 &\le \left[\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + m\zeta(r)\right]^2 \\ &\quad +1 - |\nabla r|^2 + a(r)^2 \\ &\le C\Big(\zeta(r)^2 + 1 - |\nabla r|^2 + a(r)^2\Big) \end{aligned} \tag{4.6}$$

for some absolute constant C. Note that, in both our pointwise or integral pinching assumptions on N, by Proposition 2.1 it holds $\zeta(s) \to 0$ as $s \to +\infty$. Set

$$F(t) \doteq \sup_{\sigma \in [t-1,+\infty)} [a(\sigma)^2 + \zeta(\sigma)^2],$$

and note that $F(t) \to 0$ monotonically as $t \to +\infty$. Integrating (4.6) we get the existence of $C > 0$ independent of s,t such that

$$\begin{aligned} &\int_{A_{t-1,S}} \left[\left(\Delta r - \frac{v_k'}{v_k}\right)^2 + 1 - |\nabla r|^2 + a(r)^2\right] \mathrm{d}\mu_k \\ &\quad \le C\left(F(t)\int_{A_{t-1,S}} \frac{1}{v_k(r)} + \int_{A_{t-1,S}} \frac{1-|\nabla r|^2}{v_k(r)}\right). \end{aligned} \tag{4.7}$$

Using the coarea's formula and the transversality lemma, for each $0 \le a < b$

$$\mu_k(A_{a,b}) = \int_{A_{a,b}} \frac{1}{v_k(r)} = \int_a^b J\big[1+T\big], \qquad \int_{A_{a,b}} \frac{1-|\nabla r|^2}{v_k(r)} = \int_a^b JT, \quad (4.8)$$

where J and T are defined, respectively, in (3.2) and (3.15). Summarizing, in view of (4.7) and (4.8) we deduce from (4.4) the following inequalities:

$$\begin{aligned} \frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} \quad \le \quad & C\left(\frac{1}{\int_t^s J\big[1+T\big]}\left[F(t)\int_{t-1}^S J\big[1+T\big] + \int_{t-1}^S JT\right]\right. \\ & \left. + \frac{\int_s^S J\big[1+T\big]}{(S-s)^2\int_t^s J\big[1+T\big]} + \frac{\int_{t-1}^t J\big[1+T\big]}{\int_t^s J\big[1+T\big]}\right) \doteq \mathcal{Q}(t,s). \end{aligned} \quad (4.9)$$

If we can guarantee that

$$\liminf_{t\to+\infty}\liminf_{s\to+\infty} \frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} = 0, \quad (4.10)$$

then we are able to construct a sequence of approximating eigenfunctions for λ as follows: fix $\varepsilon > 0$. By (4.10) there exists a divergent sequence $\{t_i\}$ such that, for $i \ge i_\varepsilon$,

$$\liminf_{s\to+\infty} \frac{\|\Delta u_{t_i,s} + \lambda u_{t_i,s}\|_2^2}{\|u_{t_i,s}\|_2^2} < \varepsilon/2.$$

For $i = i_\varepsilon$, pick then a sequence $\{s_j\}$ realizing the liminf. For $j \ge j_\varepsilon(i_\varepsilon, \varepsilon)$

$$\|\Delta u_{t_i,s_j} + \lambda u_{t_i,s_j}\|_2^2 < \varepsilon \|u_{t_i,s_j}\|_2^2, \quad (4.11)$$

Writing $u_\varepsilon \doteq u_{t_{i_\varepsilon}, s_{j_\varepsilon}}$, by (4.11) from the set $\{u_\varepsilon\}$ we can extract a sequence of approximating eigenfunctions for λ, concluding the proof that $\lambda \in \sigma(M)$. To show (4.10), by (4.9) it is enough to prove that

$$\liminf_{t\to+\infty}\liminf_{s\to+\infty} \mathcal{Q}(t,s) = 0. \tag{4.12}$$

Suppose, by contradiction, that (4.12) were not true. Then, there exists a constant $\delta > 0$ such that, for each $t \geq t_\delta$, $\liminf_{s\to+\infty} \mathcal{Q}(t,s) \geq 2\delta$, and thus for $t \geq t_\delta$ and $s \geq s_\delta(t)$

$$\begin{aligned} \delta \int_t^s J\big[1+T\big] \quad &\leq \quad F(t)\int_{t-1}^S J\big[1+T\big] + \int_{t-1}^S JT + \int_s^S \frac{J\big[1+T\big]}{(S-s)^2} \\ &\quad + \int_{t-1}^t J\big[1+T\big], \end{aligned} \tag{4.13}$$

and rearranging

$$\begin{aligned} \delta \int_t^s J\big[1+T\big] \quad &\leq \quad (F(t)+1)\int_{t-1}^S J\big[1+T\big] - \int_{t-1}^S J + \int_s^S \frac{J\big[1+T\big]}{(S-s)^2} \\ &\quad + \int_{t-1}^t J\big[1+T\big]. \end{aligned} \tag{4.14}$$

We rewrite the above integrals in order to make $\Theta(s)$ appear. Integrating by parts and using again the coarea's formula and the transversality lemma,

$$\begin{aligned} \int_a^b J\big[1+T\big] \quad &= \quad \int_{A_{a,b}} \frac{1}{v_k(r)} = \int_a^b \frac{1}{v_k(\sigma)}\left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma \\ &= \quad \int_a^b \frac{\big(V_k(\sigma)\Theta(\sigma)\big)'}{v_k(\sigma)} \mathrm{d}\sigma \\ &= \quad \frac{V_k(b)}{v_k(b)}\Theta(b) - \frac{V_k(a)}{v_k(a)}\Theta(a) + \int_a^b \frac{V_k v_k'}{v_k^2}\Theta. \end{aligned} \tag{4.15}$$

To deal with the term containing the integral of J alone in (4.14), we use the inequality $J(s) \geq \Theta(s)$ coming from the monotonicity formulae in Proposition 3.1. This passage is crucial for us to conclude. Inserting (4.15) and $J \geq \Theta$ into (4.14) we get

$$\begin{aligned}
&(F(t)+1)\left(\frac{V_k(S)}{v_k(S)}\Theta(S)-\frac{V_k(t-1)}{v_k(t-1)}\Theta(t-1)+\int_{t-1}^{S}\frac{V_kv_k'}{v_k^2}\Theta\right)\\
&-\int_{t-1}^{S}\Theta+\frac{1}{(S-s)^2}\left[\frac{V_k(S)}{v_k(S)}\Theta(S)-\frac{V_k(s)}{v_k(s)}\Theta(s)+\int_{s}^{S}\frac{V_kv_k'}{v_k^2}\Theta\right]\\
&+\frac{V_k(t)}{v_k(t)}\Theta(t)-\frac{V_k(t-1)}{v_k(t-1)}\Theta(t-1)+\int_{t-1}^{t}\frac{V_kv_k'}{v_k^2}\Theta\\
&\qquad\qquad \geq \quad \delta\frac{V_k(s)}{v_k(s)}\Theta(s)-\delta\frac{V_k(t)}{v_k(t)}\Theta(t)+\delta\int_{t}^{s}\frac{V_kv_k'}{v_k^2}\Theta.
\end{aligned} \tag{4.16}$$

The idea to reach the desired contradiction is to prove that, as a consequence of (4.16),

$$\int_{t-1}^{S}\Theta \tag{4.17}$$

(hence, $\Theta(S)$) must grow faster as $S \to +\infty$ than the bound in (1.9). To do so, we need to simplify (4.16) in order to find a suitable differential inequality for (4.17).

We first observe that, both for $k>0$ and for $k=0$, there exists an absolute constant $\hat{c}$ such that $\hat{c}^{-1} \leq V_kv_k'/v_k^2 \leq \hat{c}$ on $[1,+\infty)$. Furthermore, by the monotonicity of Θ,

$$\int_{s}^{S}\frac{V_kv_k'}{v_k^2}\Theta \leq \hat{c}(S-s)\Theta(S). \tag{4.18}$$

Next, we deal with the two terms in the left-hand side of (4.16) that involve (4.17):

$$\begin{aligned}
(F(t)+1)\int_{t-1}^{S}\frac{V_kv_k'}{v_k^2}\Theta-\int_{t-1}^{S}\Theta &= F(t)\int_{t-1}^{S}\frac{V_kv_k'}{v_k^2}\Theta+\int_{t-1}^{S}\frac{V_kv_k'-v_k^2}{v_k^2}\Theta\\
&\leq \hat{c}F(t)\int_{t-1}^{S}\Theta+\int_{t-1}^{S}\frac{V_kv_k'-v_k^2}{v_k^2}\Theta.
\end{aligned}$$

The key point is the following relation:

$$\frac{V_k(s)v_k'(s)-v_k(s)^2}{v_k(s)^2}\begin{cases} = -1/m & \text{if } k=0;\\ \to 0 \text{ as } s\to+\infty, & \text{if } k>0.\end{cases} \tag{4.19}$$

Define

$$\omega(t)\doteq \sup_{[t-1,+\infty)}\frac{V_kv_k'-v_k^2}{v_k^2}, \qquad \chi(t)\doteq \hat{c}F(t)+\omega(t).$$

Again by the monotonicity of Θ,

$$\begin{aligned}(F(t)+1)\int_{t-1}^{S}\frac{V_kv_k'}{v_k^2}\Theta-\int_{t-1}^{S}\Theta \;&\leq\; \left[\hat{c}F(t)+\omega(t)\right]\int_{t-1}^{S}\Theta\\ &=\; \chi(t)\int_{t-1}^{S}\Theta\\ &\leq\; \chi(t)\Theta(t)+\chi(t)\int_{t}^{S}\Theta.\end{aligned} \tag{4.20}$$

For simplicity, hereafter we collect all the terms independent of s in a function that we call $h(t)$, which may vary from line to line. Inserting (4.18) and (4.20) into (4.16) we infer

$$\begin{aligned}&\left[\left(F(t)+1+\frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)}+\frac{\hat{c}}{S-s}\right]\Theta(S)+\chi(t)\int_t^S\Theta\\ &\geq h(t)+\left(\delta+\frac{1}{(S-s)^2}\right)\frac{V_k(s)}{v_k(s)}\Theta(s)+\delta\hat{c}^{-1}\int_t^s\Theta.\end{aligned} \tag{4.21}$$

Summing $\delta\hat{c}^{-1}(S-s)\Theta(S)$ to the two sides of the above inequality, using the monotonicity of Θ and getting rid of the term containing $\Theta(s)$ we obtain

$$\left[\left(F(t)+1+\frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)}+\frac{\hat{c}}{S-s}+\delta\hat{c}^{-1}(S-s)\right]\Theta(S)+\chi(t)\int_t^S\Theta$$
$$\geq h(t)+\delta\hat{c}^{-1}\int_t^S\Theta. \tag{4.22}$$

Using (4.19), the definition of $\chi(t)$ and the properties of $\omega(t)$, $F(t)$, we can choose t_δ sufficiently large to guarantee that

$$\delta\hat{c}^{-1}-\chi(t)\geq c_k \doteq \begin{cases} \frac{1}{m}+\frac{\delta\hat{c}^{-1}}{2} & \text{if } k=0,\\ \frac{\delta\hat{c}^{-1}}{2} & \text{if } k>0,\end{cases} \tag{4.23}$$

hence

$$\left[\left(F(t)+1+\frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)}+\frac{\hat{c}}{S-s}+\delta\hat{c}^{-1}(S-s)\right]\Theta(S)\geq h(t)+c_k\int_t^S\Theta. \tag{4.24}$$

We now specify $S(s)$ depending on whether $k>0$ or $k=0$.

The case $k>0$.
We choose $S \doteq s+1$. In view of the fact that V_k/v_k is bounded above on $\mathbb{R}^+$, (4.24) becomes

$$\bar{c}\Theta(s+1)\geq h(t)+c_k\int_t^{s+1}\Theta\geq\frac{c_k}{2}\int_t^{s+1}\Theta, \tag{4.25}$$

for some $\bar{c}$ independent of t,s. Note that the last inequality is satisfied provided $s\geq s_\delta(t)$ is chosen to be sufficiently large, since the monotonicity of Θ implies that $\Theta\notin L^1(\mathbb{R}^+)$. Integrating and using again the monotonicity of Θ, we get

$$(s+1-t)\Theta(s+1)\geq\int_t^{s+1}\Theta\geq\left[\int_t^{s_0+1}\Theta\right]\exp\left\{\frac{c_k}{2\bar{c}}(s-s_0)\right\},$$

hence $\Theta(s)$ grows exponentially. Ultimately, this contradicts our assumption (1.9).

The case $k=0$.
We choose $S \doteq s+\sqrt{s}$. Since $V_k(S)/v_k(S) = S/m$, from (4.24) we infer

$$\left[\left(F(t)+1+\frac{1}{s}\right)\frac{S}{m}+\frac{\hat{c}}{\sqrt{s}}+\delta\hat{c}^{-1}\sqrt{s}\right]\Theta(S) \ge h(t)+c_k\int_t^S\Theta. \tag{4.26}$$

Using the expression of c_k and the fact that $F(t)\to 0$, up to choosing t_δ and then $s_\delta(t)$ large enough we can ensure the validity of the following inequality:

$$\left[\left(F(t)+1+\frac{1}{s}\right)\frac{S}{m}+\frac{\hat{c}}{\sqrt{s}}+\delta\hat{c}^{-1}\sqrt{s}\right] < \left[\frac{1}{m}+\frac{\delta\hat{c}^{-1}}{4}\right]S = \left[c_k-\frac{\delta\hat{c}^{-1}}{4}\right]S$$

for $t\ge t_\delta$ and $s\ge s_\delta(t)$. Plugging into (4.24), and using that $\Theta\notin L^1(\mathbb{R}^+)$,

$$S\Theta(S) \ge h(t)+\frac{c_k}{c_k-\delta\hat{c}^{-1}/4}\int_t^S\Theta \ge (1+\varepsilon)\int_t^S\Theta,$$

for a suitable $\varepsilon>0$ independent of t,S, and provided that $S\ge s_\delta(t)$ is large enough. Integrating and using again the monotonicity of Θ,

$$S\Theta(S)\ge (S-t)\Theta(S)\ge \int_t^S\Theta \ge \left[\int_t^{S_0}\Theta\right]\left(\frac{S}{S_0}\right)^{1+\varepsilon},$$

hence $\Theta(S)$ grows polynomially at least with power ε, contradicting (1.9).
Concluding, both for $k>0$ and for $k=0$ assuming (4.13) leads to a contradiction with our assumption (1.9), hence (4.10) holds, as required.

5 Proof of Theorem 2

We first show that φ is proper and that M is diffeomorphic to the interior of a compact manifold with boundary. Both the properties are consequence of the following lemma due to [6], which improves on [1], [20], [10], [4].

Lemma 5.1. *Let $\varphi : M^m \to N^n$ be an immersed submanifold into an ambient manifold N with a pole and suppose that N satisfies* (1.8) *for some $k \geq 0$. Denote by $B_s = \{x \in M;\ \rho(x) \leq s\}$ the intrinsic ball on M. Assume that*

$$\begin{aligned} &(i) \qquad \limsup_{s\to+\infty} s\|II\|_{L^\infty(\partial B_s)} < 1 \qquad \text{if } k = 0 \text{ in } (1.8)\text{, or} \\ &(ii) \qquad \limsup_{s\to+\infty} \|II\|_{L^\infty(\partial B_s)} < \sqrt{k} \qquad \text{if } k > 0 \text{ in } (1.8). \end{aligned} \tag{5.1}$$

Then, φ is proper and there exists $R > 0$ such that $|\nabla r| > 0$ on $\{r \geq R\}$, where r is the extrinsic distance function. Consequently, the flow

$$\Phi : \mathbb{R}^+ \times \{r = R\} \to \{r \geq R\}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(x) = \frac{\nabla r}{|\nabla r|^2}\big(\Phi_s(x)\big) \tag{5.2}$$

is well defined, and M is diffeomorphic to the interior of a compact manifold with boundary.

The properness of φ enables us to apply Proposition 3.2. Therefore, to show that $\Theta(+\infty) < +\infty$ it is enough to check that

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \frac{\int_{\Gamma_s}\left[|\nabla r|^{-1} - |\nabla r|\right]}{\int_{\Gamma_s}|\nabla r|} \in L^1(+\infty). \tag{5.3}$$

To achieve (5.3), we need to bound from above the rate of approaching of $|\nabla r|$ to 1 along the flow Φ in Lemma 5.1. We begin with the following

Lemma 5.2. *Suppose that N has a pole and radial sectional curvature satisfying* (1.8), *and that $\varphi : M^m \to N^n$ is a proper minimal immersion such that $|\nabla r| > 0$ outside of some compact set $\{r \le R\}$. Let Φ denote the flow of $\nabla r/|\nabla r|^2$ as in* (5.2) *and let $\gamma : [R, +\infty) \to M$ be a flow line starting from some $x_0 \in \{r = R\}$. Then, along γ,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{sn}_k(r)\sqrt{1-|\nabla r|^2}\Big) \le \mathrm{sn}_k(r)|II(\gamma(s))| \tag{5.4}$$

The above lemma relates the behaviour of $|\nabla r|$ to that of the second fundamental form. The next result makes this relation explicit in the two cases considered in Theorem 1.2.

Proposition 5.1. *In the assumptions of the above proposition, suppose further that either*

$$\begin{array}{lll} (i) & \|II\|_{L^\infty(\partial B_s)} \le \dfrac{C}{s\log^{\alpha/2} s} & \textit{if } k = 0 \textit{ in } (1.8), \textit{ or} \\ (ii) & \|II\|_{L^\infty(\partial B_s)} \le \dfrac{C}{\sqrt{s}\log^{\alpha/2} s} & \textit{if } k > 0 \textit{ in } (1.8). \end{array} \tag{5.5}$$

for $s \ge 1$ and some constants $C > 0$ and $\alpha > 0$. Here, ∂B_s is the boundary of the intrinsic ball $B_s(o)$. Then, $|\nabla r|(\gamma(s)) \to 1$ as s diverges, and if $s > 2R$ and R is sufficiently large,

$$\begin{array}{ll} \textit{in the case } (i), & 1 - |\nabla r(\gamma(s))|^2 \le \dfrac{\hat{C}}{\log^\alpha s} \\ \textit{in the case } (ii), & 1 - |\nabla r(\gamma(s))|^2 \le \dfrac{\hat{C}}{s\log^\alpha s} \end{array} \tag{5.6}$$

for some constant $\hat{C}$ depending on R.

We are now ready to conclude the proof of Theorem 1.2 by showing that M has finite density or, equivalently, that (5.3) holds.

Let $\eta(s)$ be either

$$\frac{1}{\log^{\alpha} s} \quad \text{when } k=0, \text{ or } \frac{1}{s\log^{\alpha} s} \quad \text{when } k>0, \tag{5.7}$$

where $\alpha > 1$ and C is a large constant. In our assumptions, we can apply Lemma 5.2 and Proposition 5.1 to deduce, according to (5.6), that, for large enough R,

$$1-|\nabla r(\gamma(s))|^2 \le C\eta(s) \qquad \text{on } (R,+\infty),$$

where $\gamma(s)$ is a flow curve of Φ in (5.2) and $C = C(R)$ is a large constant. In particular, $|\nabla r(\gamma(s))| \to 1$ as $s \to +\infty$. We therefore deduce the existence of a constant $C_2(R) > 0$ such that

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \frac{\int_{\Gamma_s}\left[|\nabla r|^{-1}-|\nabla r|\right]}{\int_{\Gamma_s}|\nabla r|} \le C\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\eta(s)\frac{\int_{\Gamma_s}|\nabla r|^{-1}}{\int_{\Gamma_s}|\nabla r|} \le C_2\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\eta(s).$$

In both our cases $k=0$ and $k>0$, since $\alpha > 1$ it is immediate to check that $\mathrm{sn}_k'\eta/\mathrm{sn}_k \in L^1(+\infty)$, proving (5.3).

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Translation Hypersurfaces with Constant Scalar Curvature into the Euclidean Space

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Dedicated to Professor Barnabé P. Lima on the occasion of his 60th birthday.
"To the master, with affection".

1 Minimal Surfaces

In 1760, Lagrange [1] proposed the following problem: *Given a closed curve C (without self-intersections), find the minimum area surface that has this curve as boundary.* Lagrange presented this problem as a mere example of a method he developed to find curves or surfaces that would minimize certain amounts such as area, length, energy, etc. These methods today constitute the so-called Calculus of Variations.

Applying the method developed by Lagrange, we conclude that if there exists a surface S of minimum area with boundary C, then $H = 0$ (where H denotes the mean curvature of S). Therefore, the minimal area surfaces are the minimal surfaces, that is, the surfaces whose mean curvature is zero at all points ($H = 0$). However, Lagrange gave no examples of minimal surfaces except the trivial example of the plane.

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In fact, to find examples of surfaces with $H = 0$ is not, in principle, an easy task. Even for the simplest case of surfaces that are graphs $z = f(x,y)$ of differentiable functions (which was the case treated by Lagrange). In this case, the condition $H = 0$ is equivalent to the equation

$$(1+f_y^2)f_{xx} + 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 0. \tag{1.1}$$

The linear functions $f(x,y) = ax + by + c$, where $a, b, c \in \mathbb{R}$, are solutions of this equation.

We note that the definition of mean curvature was not known in Lagrange time. In fact, the main curvatures λ_1 and λ_2 were still not defined, which were introduced by Euler in a paper published in the same year as Lagrange's work. What Lagrange did was to use the method of variations for surfaces in the form $z = f(x,y)$ and to obtain that the equation (1.1) was a necessary condition for a surface to have minimum area.

Sixteen years after Lagrange obtained the equation (1.1), Meusnier [2] showed that it was equivalent to the fact that $\lambda_1 + \lambda_2 = 0$, and obtained two nontrivial solutions of this equation. Namely: the *catenoid*, the only minimal surface area of rotation, unless translations and dilations; The *helicoid*, the only (except the plane) minimal ruled surface.

For a long time, the plane, the catenoid and the helicoid were the only known examples of minimal surfaces. In 1835, Scherk [3] obtained a new example by introducing into the equation (1.1) the condition that the variables could be separated. More precisely, Scherk assumed that $f(x,y) = g(x) + h(y)$. With this condition, the partial derivatives are replaced by ordinary

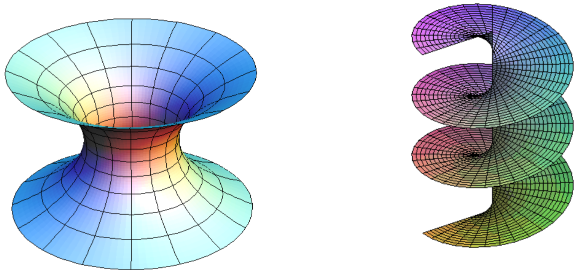

Figure 1: Catenoid and Helicoid.

derivatives and the equation takes the form

$$(1+\dot{h}^2(y))\ddot{g}(x)+(1+\dot{g}^2(x))\ddot{h}(y)=0,$$

that is

$$\frac{\ddot{g}(x)}{1+\dot{g}^2(x)}=-\frac{\ddot{h}(y)}{1+\dot{h}^2(y)}=\text{constant},$$

whose integration gives us,

$$f(x,y)=\frac{1}{a}\ln\left|\frac{\cos(ay)}{\operatorname{sen}(ax)}\right|$$

where a is a nonzero constant. Such surfaces are known as the Scherk surfaces.

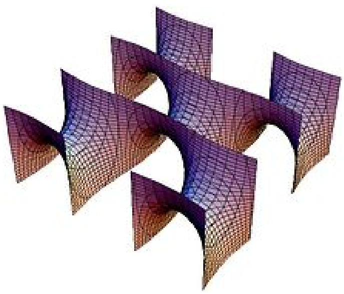

Figure 2: Scherk Surface.

2 Generalizations of the Scherk Surfaces

Along the years, Scherk surfaces in the Euclidean 3-space were generalized. Before we present some generalizations, consider the following definition.

Definition 2.1. *In the Euclidean space $\mathbb{R}^3$, a surface is called a translation surface if it is parametrized as*

$$\Psi : U \subset \mathbb{R}^2 \to \mathbb{R}^3;\ \Psi(x,y) = (x,y,g(x)+h(y)),$$

where f and g are smooth functions.

That is, M^2 can be thought as a composition of plane curves given by graphs, that is: denote by $\alpha(x) = (x,0,g(x))$ and $\beta(y) = (0,y,h(y))$. For $p \in \mathbb{R}^3$ denote by $L_p : \mathbb{R}^3 \to \mathbb{R}^3$, the translation through p, given by $L_p(q) = p+q$. Then, the map Ψ above is given by

$$\Psi(x,y) = L_{\alpha(x)}(\beta(y)) = \alpha(x)+\beta(y).$$

The natural generalization to be considered for Scherk surfaces (minimal translational surfaces) are translational surfaces with constant mean curvature. In this case, Liu [4] obtained the classification of such surfaces.

In another generalization, the Euclidean space $\mathbb{R}^3$ was replaced with other space, usually a three-dimensional Lie group, and the notion of the translation surface was carefully adapted by use of the group operation. For some of these generalizations, see, e.g. [5], [6] e [7].

On the other hand, one can consider the translational surfaces in $\mathbb{R}^3$ expressed in parametric form as

$$\Psi(s,t) = \alpha(s) + \beta(t),$$

where $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$ are regular curves satisfying the condition $\alpha'(s) \times \beta'(t) \neq 0$. Dillen *et al.* [8] proved that: *There are no minimal surfaces in the three dimensional Euclidean space, defined as the sum of a planar curve and a space curve*. Later, López and Perdomo [9] obtained a characterization of the minimal translational surfaces when α e β are not planar.

Finally, translational surfaces were generalized to translational hypersurfaces in an Euclidean space of arbitrary dimension. In the next section, we'll go into more detail.

3 Translation Hypersurfaces

In this section we will deal in more detail with the translational hypersurfaces in an Euclidean space of arbitrary dimension, direction in which we have obtained our results.

Definition 3.1. *We say that a hypersurface M^n of the Euclidean space $\mathbb{R}^{n+1}$ is a translation hypersurface if it is the graph of a function given by*

$$F(x_1, \ldots, x_n) = f_1(x_1) + \ldots + f_n(x_n)$$

where $(x_1, \ldots, x_n)$ are cartesian coordinates and f_i is a smooth function of one real variable for $i = 1, \ldots, n$.

Dillen, Verstraelen and Zafindratafa [10] obtained a classification of minimal translation hypersurfaces of the $(n+1)$-dimensional Euclidean space. A classification of translation hypersurfaces with constant mean curvature in $(n+1)$-dimensional Euclidean space was made by Chen, Sun and Tang [11].

Now, let $M^n \subset \mathbb{R}^{n+1}$ be an oriented hypersurface and $\lambda_1, \ldots, \lambda_n$ denote the principal curvatures of M^n. We can consider similar problems related with the r-th elementary symmetric polynomials, S_r, given by $S_r = \sum \lambda_{i_1} \cdots \lambda_{i_r}$, where $r = 1, \ldots, n$ and $1 \leq i_1 < \cdots < i_r \leq n$. In particular, S_1 is the mean curvature, S_2 the scalar curvature and S_n the Gauss-Kronecker curvature, up to normalization factors. A very useful relationship involving the various S_r is given by the next proposition. This result will play a central role along this paper.

Proposition 3.1 (Caminha, 2006 [12]). *Let $n > 1$ be an integer, and $\lambda_1, \ldots, \lambda_n$ be real numbers. Define, for $0 \leq r \leq n$, $S_r = S_r(\lambda_1)$ as above, and set $H_r = H_r(\lambda_i) = \binom{n}{r}^{-1} S_r(\lambda_i)$*

(a) For $1 \leq r \leq n$, one has $H_r^2 \geq H_{r-1}H_{r+1}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1} \neq 0$ in this case, then $\lambda_1 = \ldots = \lambda_n$.

(b) If $H_1, H_2, \ldots H_r > 0$ for some $1 < r \leq n$, then

$$H_1 \geq \sqrt{H_2} \geq \sqrt[3]{H_3} \geq \cdots \geq \sqrt[r]{H_r}.$$

Moreover, if equality happens for some $1 \leq j < r$, then $\lambda_1 = \ldots = \lambda_n$.

(c) If for some $1 \leq r < n$, one has $H_r = H_{r+1} = 0$, then $H_j = 0$ for all $r \leq j \leq n$. In particular, at most $r-1$ of the λ_i are different from zero.

In this paper, we obtain a complete classification of translation hypersurfaces of $\mathbb{R}^{n+1}$ with zero scalar curvature. More specifically, we prove the following

Theorem 3.1. *Let $M^n\,(n \geq 3)$ be a translation hypersurface in $\mathbb{R}^{n+1}$. Then M^n has zero scalar curvature if, and only if, it is congruent to the graph of the following functions*

- *$F(x_1,\ldots,x_n) = \sum_{i=1}^{n-1} a_i x_i + f_n(x_n) + b$, on $\mathbb{R}^{n-1} \times J$, for some interval J, and $f_n : J \subset \mathbb{R} \to \mathbb{R}$ is a smooth function. Which defines, after a suitable linear change of variables, a vertical cylinder, and*

- *A generalized periodic Enneper hypersurface given by*

$$\begin{aligned} F(x_1,\ldots,x_n) &= \sum_{i=1}^{n-3} a_i x_i + \frac{\sqrt{\beta}}{a} \ln \left| \frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}x_n + c\right)}{\cos(a\sqrt{\beta}x_{n-2} + a_0)} \right| \\ &+ \frac{\sqrt{\beta}}{b} \ln \left| \frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}x_n + c\right)}{\cos(b\sqrt{\beta}x_{n-1} + b_0)} \right| + d \end{aligned}$$

on $\mathbb{R}^{n-3} \times I_1 \times I_2 \times I_3$, where $a, a_1, \ldots, a_{n-3}, b, b_0, c, d$ are real constants with $a, b, a+b \neq 0$, $\beta = 1 + \sum_{i=1}^{n-3} a_i^2$ and I_1, I_2, I_3 are the opens intervals defined, respectively, by the conditions $|a\sqrt{\beta}x_{n-2} + a_0| < \pi/2$, $|b\sqrt{\beta}x_{n-1} + b_0| < \pi/2$ and $\left|-\frac{ab}{a+b}\sqrt{\beta}x_n + c\right| < \pi/2$.

In [13], T. Okayasu used the method of equivariant geometry to construct the first example of a complete hypersurface with constant negative scalar curvature in $\mathbb{R}^4$ $(n \geq 4)$. And, in [14], T. Okayasu constructed a new family of noncompact complete hypersurfaces with constant positive scalar curvature in the Euclidean space. However, these examples do not define entire graphs. Moreover, we don't know whether there exists an entire graph on $\mathbb{R}^n$ with constant negative scalar curvature and, in [15], Alencar *et al* showed in Corollary 4.1 that: *Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature*. In this work, we classify the translation hypersurfaces of $\mathbb{R}^{n+1}$ with constant scalar curvature and we obtain a result that endorses the fact proved by Alencar *et al*. More precisely,

In this work, we classified the translational hypersurfaces of $\mathbb{R}^{n+1}$ with constant scalar curvature and obtained a result that endorses the fact proved by Alencar *et al*. More precisely,

Theorem 3.2. *Any translation hypersurface in* $\mathbb{R}^{n+1}\,(\geq 3)$ *with constant scalar curvature must have zero scalar curvature.*

4 Proof of the Results

For a better understanding of the results which we will prove in this section, we first introduce some notations, definitions and basic facts.

Let $\overline{M}^{n+1}$ be a connected Riemannian manifold. In the remainder of this paper, we will be concerned with isometric immersions, $\Psi : M^n \to \overline{M}^{n+1}$, from a connected, n-dimensional orientable Riemannian manifold, M^n, into $\overline{M}^{n+1}$. We fix an orientation of M^n, by choosing a globally defined unit normal vector field, N, on M. Denote by A, the corresponding shape operator. At each $p \in M$, A restricts to a self-adjoint linear map $A_p : T_pM \to T_pM$. For each $1 \leq r \leq n$, let $S_r : M^n \to \mathbb{R}$ be the smooth function such that $S_r(p)$ denotes the r-th elementary symmetric function on the eigenvalues of A_p, which can be defined by the identity

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k}.$$

where $S_0 = 1$ by definition. If $p \in M^n$ and $\{e_l\}$ is a basis of T_pM, given by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_l\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on $X_1, \dots, X_n$. Consequently,

$$S_r = \sum_{1 \le i_1 < \cdots < i_r \le n} \lambda_{i_1} \cdots \lambda_{i_r}, \text{ where } r = 1, \dots, n.$$

When the ambient space $\overline{M}^{n+1}$ has constant sectional curvature, c, Gauss equation allows one to check, immediately, that the scalar curvature R of M^n relates to S_2 in the following form:

$$n(n-1)(R-c) = 2S_2.$$

When $c = 0$, we obtain a particular relationship between the scalar curvature and the second symmetric function of the principal curvatures: $n(n-1)R = 2S_2$. Hence we conclude that the scalar curvature is constant and equal to zero if, and only if, S_2 vanishes identically.

Related to the curvature S_2, we also present in the next proposition, a useful result that will be used later. In order to do so, consider the following notation: for $1 \le i_1, \dots, i_q, j_1, \dots, j_q \le n$, the Kronecker symbol $\delta\binom{i_1 \cdots i_q}{j_1 \cdots j_q}$ has the value $+1$ (respectively -1) if $i_1, \dots, i_q$ are distinct and $(j_1 \cdots j_q)$ is an even (respectively, an odd) permutation of $(i_1 \cdots i_q)$. Otherwise, it has value 0.

Proposition 4.1 (Reilly [16], Prop. 1.2). *Suppose that, relative to some basis of the vector space V, the self-adjoint linear map $A: V \to V$ has matrix A_i^j. Then,*

$$S_q(A) = \frac{1}{q!} \sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} A_{i_1}^{j_q} \cdots A_{i_q}^{j_q},$$

where S_q is the q-th elementary symmetric polynomial of the eigenvalues of the linear map A.

Now, suppose that $M^n \hookrightarrow \mathbb{R}^{n+1}$ is the graph $x_{n+1} = F(x_1, \ldots, x_n)$. We denote the natural parametrization of M^n by

$$\Psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n)) = \sum_{j=1}^{n} x_j e_j + F(x_1, \ldots, x_n) e_{n+1}.$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the j-th place), and the partial derivatives $\frac{\partial F}{\partial x_j}, \frac{\partial^2 F}{\partial x_j \partial x_k}, \cdots$, by $F_j, F_{jk}, \cdots$. We recall the standard calculations in a single proposition.

Proposition 4.2 (Reilly [16], Proposition 3.1)**.** *If $M^n \to \mathbb{R}^{n+1}$ is the graph $x_{n+1} = F(x_1, \ldots, x_n)$ and $W = \sqrt{1 + F_1^2 + \ldots + F_n^2}$, then:*

(a) The natural frame field on M^n is $\Psi_1, \ldots, \Psi_n$ where $\Psi_i = \frac{\partial \Psi}{\partial x_i} = e_i + F_i e_{n+1}$ and the unit normal is

$$N = -\sum_{j=1}^{n} \frac{F_j}{W} e_j + \frac{1}{W} e_{n+1}.$$

(b) The matrix of the shape operator is

$$b_i^j = \frac{F_{ij}}{W} - \sum_{k=1}^{n} \frac{F_{ik} F_k F_j}{W^3}.$$

To conclude this section, we present in the next proposition the equation for the scalar curvature of a translation hypersurface in $\mathbb{R}^{n+1}$.

Proposition 4.3. *Let M^n be a translation hypersurface immersed in $\mathbb{R}^{n+1}$ parametrized by*

$$\Psi : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1} : \Psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n))$$

where $(x_1,\ldots,x_n)$ *are cartesian coordinates,* $F(x_1,\ldots,x_n) = \sum_{i=1}^{n} f_i(x_i)$ *and* f_i *is a smooth function of one real variable. Then,*

$$S_2 = \frac{1}{W^4} \sum_{1\le i<j\le n} \ddot{f}_i \ddot{f}_j \Big(1 + \sum_{\substack{1\le k\le n \\ k\ne i,j}} \dot{f}_k^2\Big). \tag{4.1}$$

Proof. It follows from Proposition 4.2 and the fact that $F_{ij} = 0$, when $i \ne j$, that the matrix of the shape operator has the following expression

$$b_i^j = \frac{1}{W^3}(F_{ij}W^2 - F_{ii}F_iF_j).$$

Now, from Proposition 4.1 we obtain

$$S_2 = \sum_{1\le i_1<i_2\le n} \delta\begin{pmatrix} i_1\, i_2 \\ j_1\, j_2 \end{pmatrix} A_{i_1}^{j_1} \cdot A_{i_2}^{j_2} = \sum_{1\le i<j\le n} (b_i^i b_j^j - b_i^j b_j^i).$$

Again from Proposition 4.2, we get

$$\begin{aligned} b_i^i b_j^j - b_i^j b_j^i &= \frac{1}{W^6}\left[\ddot{f}_i(W^2 - \dot{f}_i^2)\ddot{f}_j(W^2 - \dot{f}_j^2) - (-\ddot{f}_i\dot{f}_i\dot{f}_j)(-\ddot{f}_j\dot{f}_j\dot{f}_i)\right] \\ &= \frac{1}{W^6}\ddot{f}_i\ddot{f}_j\left[(W^2 - \dot{f}_i^2)(W^2 - \dot{f}_j^2) - \dot{f}_i^2\dot{f}_j^2\right] \\ &= \frac{1}{W^4}\ddot{f}_i\ddot{f}_j(W^2 - \dot{f}_i^2 - \dot{f}_j^2). \end{aligned}$$

This completes the proof of the proposition. □

In order to prove Theorem 3.1 we need the following lemma.

Lemma 4.1. *Let* f, g, *and* h *be smooth functions of one real variable satisfying the differential equation*

$$\ddot{f}(x)\ddot{g}(y)(\beta + \dot{h}^2(z)) + \ddot{f}(x)\ddot{h}(z)(\beta + \dot{g}^2(y)) + \ddot{g}(y)\ddot{h}(z)(\beta + \dot{f}^2(x)) = 0,$$

where β is a positive real constant. If $\ddot{f} \neq 0$, $\ddot{g} \neq 0$ and $\ddot{h} \neq 0$, then

$$
\begin{aligned}
f(x)+g(y)+h(z) &= \frac{\sqrt{\beta}}{a}\ln\left|\frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}z+c\right)}{\cos(a\sqrt{\beta}x+a_0)}\right| \\
&\quad +\frac{\sqrt{\beta}}{b}\ln\left|\frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}z+c\right)}{\cos(b\sqrt{\beta}y+b_0)}\right|+d
\end{aligned}
$$

where a, a_0, b, b_0, c and d are real constants with $a, b, a+b \neq 0$.

Proof. The functions f, g, and h satisfy the differential equation above if and only if

$$\frac{\ddot{f}\ddot{g}}{(\beta+\dot{f}^2)(\beta+\dot{g}^2)}+\frac{\ddot{f}\ddot{h}}{(\beta+\dot{f}^2)(\beta+\dot{h}^2)}+\frac{\ddot{g}\ddot{h}}{(\beta+\dot{g}^2)(\beta+\dot{h}^2)}=0. \tag{4.2}$$

Derivatives with respect to x and y, in (4.2), lead to the equations

$$\left(\frac{\ddot{f}}{\beta+\dot{f}^2}\right)'=0 \text{ or } \left(\frac{\ddot{g}}{\beta+\dot{g}^2}\right)'=0.$$

If $\ddot{f}=a(\beta+\dot{f}^2)$ for some constant $a\neq 0$, since $\ddot{f}\neq 0$, substituting in (4.2) gives

$$\frac{\ddot{g}}{\beta+\dot{g}^2}a+\frac{\ddot{h}}{\beta+\dot{h}^2}a+\frac{\ddot{g}\ddot{h}}{(\beta+\dot{g}^2)(\beta+\dot{h}^2)}=0. \tag{4.3}$$

Now, take the derivatives of (4.3) with respect to y and z to obtain

$$\left(\frac{\ddot{g}}{\beta+\dot{g}^2}\right)'=0 \text{ or } \left(\frac{\ddot{h}}{\beta+\dot{h}^2}\right)'=0.$$

If $\ddot{g}=b(\beta+\dot{g}^2)$, for some nonzero constant b, since $\ddot{g}\neq 0$, substituting in (4.3) we get

$$ab + \frac{\ddot{h}}{\beta + \dot{h}^2}(a+b) = 0.$$

which implies that $a+b \neq 0$, since $ab \neq 0$. Integration of these ordinary differential equations implies

$$\begin{aligned}
\arctan\left(\frac{\dot{f}(x)}{\sqrt{\beta}}\right) &= a\sqrt{\beta}x + a_0 \\
\arctan\left(\frac{\dot{g}(y)}{\sqrt{\beta}}\right) &= b\sqrt{\beta}z + b_0 \\
\arctan\left(\frac{\dot{h}(z)}{\sqrt{\beta}}\right) &= -\frac{ab}{a+b}\sqrt{\beta}z + c.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x) &= -\frac{1}{a}\sqrt{\beta}\ln|\cos(a\sqrt{\beta}x + a_0)| + a_1 \\
g(x) &= -\frac{1}{b}\sqrt{\beta}\ln|\cos(b\sqrt{\beta}y + b_0)| + b_1 \\
h(x) &= \frac{a+b}{ab}\sqrt{\beta}\ln\left|\cos\left(-\frac{ab}{a+b}\sqrt{\beta}z + c\right)\right| + c_1.
\end{aligned}$$

Denoting $d = a_1 + b_1 + c_1$, we conclude the proof of the lemma. □

With this lemma at hand we can go to the proof of Theorem 3.1.

4.1 Proof of Theorem 3.1

Proof. From Proposition 4.3, we have that M^n has zero scalar curvature if, and only if,

$$\sum_{1\le i<j\le n} \ddot{f}_i \ddot{f}_j \Big(1+\sum_{\substack{1\le k\le n\\ k\ne i,j}} \dot{f}_k^2\Big)=0. \tag{4.4}$$

In order to ease the analysis, we divide the proof in four cases.

Case 1: Suppose $\ddot{f}_i(x_i)=0$, $\forall\, i=1,\dots,n-1$. In this case, we have no restrictions on the function f_n, thus

$$\Psi(x_1,\dots,x_n)=(x_1,\dots,x_n,\sum_{i=1}^{n-1} a_i x_i+f_n(x_n)+b)$$

where $a_i, b\in\mathbb{R}$ and $f_n: I\subset\mathbb{R}\to\mathbb{R}$ is a smooth function of one real variable. Note that the parametrization obtained comprise hyperplanes.

Case 2: Suppose $\ddot{f}_i(x_i)=0$, $\forall\, i=1,\dots,n-2$, then, there are constants α_i such that $\dot{f}_i=\alpha_i$. From (4.4) we have

$$\ddot{f}_{n-1}\ddot{f}_n(1+\alpha_1^2+\cdots+\alpha_{n-2}^2)=0,$$

from which we conclude that $\ddot{f}_{n-1}=0\,(\text{or}\,\ddot{f}_n=0)$ that is contained in the Case 1.

Case 3: Now suppose $\ddot{f}_i(x_i)=0$, $\forall\, i=1,\dots,n-3$ and $\ddot{f}_k(x_k)\ne 0$, for every $k=n-2,n-1,n$. Observe that if we had $\ddot{f}_k(x_k)=0$ for some $k=n-2,n-1,n$ the analysis would reduce to the Cases 1 and 2. In this case, there are constants α_i such that $\dot{f}_i=\alpha_i$ for any $1\le i\le n-3$. From (4.4) we have

$$\ddot{f}_{n-2}\ddot{f}_{n-1}(\beta+\dot{f}_n^2)+\ddot{f}_{n-2}\ddot{f}_n(\beta+\dot{f}_{n-1}^2)+\ddot{f}_{n-1}\ddot{f}_n(\beta+\dot{f}_{n-2}^2)=0$$

where $\beta=1+\sum_{k=1}^{n-3}\alpha_k^2$. Then, from Lemma 4.1 we have that

$$\sum_{k=n-2}^{n} f_k(x_k) = \frac{\sqrt{\beta}}{a}\ln\left|\frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}x_n+c\right)}{\cos(a\sqrt{\beta}x_{n-2}+a_0)}\right| + \frac{\sqrt{\beta}}{b}\ln\left|\frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}x_n+c\right)}{\cos(b\sqrt{\beta}x_{n-1}+b_0)}\right| + d$$

where a, a_0, b, b_0, c and d are real constants, and a, b, $a+b$ are nonzero.

Case 4: Finally, suppose $\ddot{f}_i(x_i) = 0$, where $1 \le i \le k$ and $n-k \ge 4$, and $\ddot{f}_i(x_i) \neq 0$ for any $i > k$. We will show that this case cannot occur. In fact, note that for any fixed $l \ge k+1$

$$\sum_{1\le i<j\le n} \ddot{f}_i\ddot{f}_j\Big(1+\sum_{\substack{1\le m\le n\\ m\neq i,j}} \dot{f}_m^2\Big) = \ddot{f}_l \sum_{\substack{1\le j\le n\\ j\neq l}} \ddot{f}_j\Big(1+\sum_{\substack{1\le m\le n\\ m\neq l,j}} \dot{f}_m^2\Big) + \sum_{\substack{1\le i<j\le n\\ i,j\neq l}} \ddot{f}_i\ddot{f}_j\Big(1+\sum_{\substack{1\le m\le n\\ m\neq i,j}} \dot{f}_m^2\Big).$$

Derivative with respect to the variable x_l $(l \ge k+1)$, in the above equality, gives

$$\dddot{f}_l \sum_{\substack{1\le j\le n\\ j\neq l}} \ddot{f}_j\Big(1+\sum_{\substack{1\le m\le n\\ m\neq l,j}} \dot{f}_m^2\Big) + 2\dot{f}_l\ddot{f}_l \sum_{\substack{1\le i<j\le n\\ i,j\neq l}} \ddot{f}_i\ddot{f}_j = 0. \tag{4.5}$$

That is, if we set

$$A_l = \sum_{\substack{1\le j\le n\\ j\neq l}} \ddot{f}_j\Big(1+\sum_{\substack{1\le m\le n\\ m\neq l,j}} \dot{f}_m^2\Big) \quad \text{and} \quad B_l = \sum_{\substack{1\le i<j\le n\\ i,j\neq l}} \ddot{f}_i\ddot{f}_j$$

then, it follows that A_l, B_l do not depend on the variable x_l and we can write

$$A_l\dddot{f}_l + 2B_l\dot{f}_l\ddot{f}_l = 0, \tag{4.6}$$

Now, we have two cases to consider: $A_l \neq 0$ and $A_l = 0$.

Case $A_l \neq 0$**:** Under this assumption, there is a constant α_l ($l = k+1,\ldots,n$) such that equation (4.6) becomes $\dddot{f}_l + 2\alpha_l \dot{f}_l \ddot{f}_l = 0$. Substituting this in identity (4.5), and using the fact that $\ddot{f}_i(x_i) \neq 0$ for $i > k$, we obtain

$$\alpha_l \sum_{\substack{1\le j\le n \\ j\neq l}} \ddot{f}_j\Big(1+\sum_{\substack{1\le m\le n \\ m\neq l,j}} \dot{f}_m^2\Big) - \sum_{\substack{1\le i<j\le n \\ i,j\neq l}} \ddot{f}_i\ddot{f}_j = 0. \tag{4.7}$$

Now, taking the derivatives of the expression (4.7) with respect to the variable x_s, for $s = k+1,\ldots,n$ and $s \neq l$, leads to

$$\alpha_l \dddot{f}_s\Big(1+\sum_{\substack{1\le m\le n \\ m\neq l,s}} \dot{f}_m^2\Big) + 2\alpha_l \dot{f}_s\ddot{f}_s \sum_{\substack{1\le j\le n \\ j\neq l,s}} \ddot{f}_j - \dddot{f}_s \sum_{\substack{1\le j\le n \\ j\neq l,s}} \ddot{f}_j = 0.$$

Again using the fact that $\dddot{f}_s + 2\alpha_s \dot{f}_s\ddot{f}_s = 0$, for any $s = k+1,\ldots,n$, and $\ddot{f}_i(x_i) \neq 0$ for $i > k$ we get the following equality

$$-\alpha_l\alpha_s\Big(1+\sum_{\substack{1\le m\le n \\ m\neq l,j}} \dot{f}_m^2\Big) + \alpha_l \sum_{\substack{1\le j\le n \\ j\neq l,s}} \ddot{f}_j + \alpha_s \sum_{\substack{1\le j\le n \\ j\neq l,s}} \ddot{f}_j = 0. \tag{4.8}$$

Finally, taking the derivative with respect to the variable x_t, on the expression (4.8), where $t = k+1,\ldots,n$, $t \neq l$ and $t \neq s$, we obtain the identity

$$\alpha_l\alpha_s + \alpha_l\alpha_t + \alpha_s\alpha_t = 0.$$

Hence we conclude that,

$$\begin{aligned} \sigma_2(\alpha_{k+1},\ldots,\alpha_n) &= 0 \\ \sigma_3(\alpha_{k+1},\ldots,\alpha_n) &= 0. \end{aligned}$$

These equalities, from Proposition 3.1, [12], imply that at most one of the constants $\alpha_l\,(l \geq k+1)$ is nonzero, suppose $\alpha_{k+1} = \cdots = \alpha_{n-1} = 0$. In this case, we have $\dddot{f}_l = 0\,(l = k+1,\ldots,n-1)$ then $\ddot{f}_l$ is constant. Now, setting $l \neq n$ in (4.7) we obtain

$$\sum_{\substack{k+1 \leq i < j \leq n \\ i \neq l \\ j \neq l}} \ddot{f}_i \ddot{f}_j = 0.$$

Which implies that $\ddot{f}_n$ is constant, and thus $\alpha_n = 0$. Now, from equation (4.7) we get

$$\sum_{\substack{\{i<j\} \subset \{l_1 < \cdots < l_{n-k-1}\} \\ \{l_1 < \cdots < l_{n-k-1}\} \subset \{k+1,\ldots,n\}}} \ddot{f}_i \ddot{f}_j = 0.$$

From which, we conclude that

$$\begin{aligned} \sigma_2(\ddot{f}_{k+1},\ldots,\ddot{f}_n) &= 0 \\ \sigma_3(\ddot{f}_{k+1},\ldots,\ddot{f}_n) &= 0. \end{aligned}$$

Thus, at most one of the functions $\ddot{f}_l\ (k+1 \leq l \leq n)$ is nonzero leading to a contradiction. Thus, it follows that Case 4 cannot occur, if $A_l \neq 0$.

Case $A_l = 0$**:** In this case, we have

$$A_l = \sum_{\substack{1 \leq j \leq n \\ j \neq l}} \ddot{f}_j \Big(1 + \sum_{\substack{1 \leq m \leq n \\ m \neq l,j}} \dot{f}_m^2\Big) = 0.$$

Consequently, derivative of A_l with respect to variable x_p, for $p = k+1,\ldots,n,\ p \neq l$, gives

$$\dddot{f}_p\Big(1+\sum_{\substack{1\le m\le n\\ m\ne l,p}} \dot{f}_m^2\Big)+2\dot{f}_p\ddot{f}_p\sum_{\substack{k+1\le j\le n\\ j\ne l,p}} \ddot{f}_j=0. \tag{4.9}$$

From (4.5), since $A_l = 0$ we have

$$2\dot{f}_l\ddot{f}_l\sum_{\substack{k+1\le i<j\le n\\ i,j\ne l}} \ddot{f}_i\ddot{f}_j=0$$

implying that

$$\sum_{\substack{k+1\le i<j\le n\\ i,j\ne l}} \ddot{f}_i\ddot{f}_j=0, \text{ since } \ddot{f}_l\ne 0.$$

Now, for $p = k+1,\ldots,n$, with $p \ne l$, the derivative of the above equation with respect to x_p gives

$$\dddot{f}_p\sum_{\substack{k+1\le j\le n\\ j\ne l,p}} \ddot{f}_j=0 \tag{4.10}$$

and for $q = k+1,\ldots,n$, with $q \ne l,p$, taking the derivative of (4.10) with respect to x_q, leads to $\dddot{f}_p\dddot{f}_q = 0$. Consequently, for at most one index, say p, one can have $\dddot{f}_p \ne 0$, and $\dddot{f}_q = 0$ for every $q = k+1,\ldots,n$, and $q \ne l,p$. Thus $\dddot{f}_p \ne 0$ together with equation (4.10) implies that the sum

$$\sum_{\substack{k+1\le j\le n\\ j\ne l,p}} \ddot{f}_j=0$$

which, on its turn inserted on equation (4.9) gives $\dddot{f}_p = 0$, contradicting $\dddot{f}_p \ne 0$. Thus, one must have $\dddot{f}_p = 0$, for every $p = k+1,\ldots,n$, and $p \ne l$. Applying these conditions on the third derivatives in (4.9) we obtain the following linear system:

$$\sum_{\substack{k+1\le j\le n\\ j\ne l,p}} \ddot{f}_j=0,\ p=k+1,\ldots,n \text{ and } p\ne l$$

which has as unique solution $\ddot{f}_j = 0$, for every $j = k+1, \ldots, n$, and $j \neq l$. This contradicts the hypothesis assumed in Case 4. Hence, $A_l = 0$ cannot occur. Since the case $A_l \neq 0$, cannot occur as well, it follows that Case 4 is not possible. This completes the proof of the theorem. $\square$

4.2 Proof of Theorem 3.2

Proof. Suppose that there is a translation hypersurface with nonzero constant scalar curvature S_2. The derivative of the expression (4.1) for S_2 with respect to x_l, gives

$$\begin{aligned} 0 = & -\frac{4\dot{f}_l\ddot{f}_l}{W^6}\sum_{1\le i<j\le n}\ddot{f}_i\ddot{f}_j\Big(1+\sum_{\substack{1\le k\le n\\ k\ne i,j}}\dot{f}_k^2\Big) \\ & + \frac{1}{W^4}\Big[\dddot{f}_l\sum_{\substack{1\le j\le n\\ j\ne l}}\ddot{f}_j\Big(1+\sum_{\substack{1\le k\le n\\ k\ne l,j}}\dot{f}_k^2\Big)+2\dot{f}_l\ddot{f}_l\sum_{\substack{1\le i<j\le n\\ i,j\ne l}}\ddot{f}_i\ddot{f}_j\Big]. \end{aligned}$$

Then,

$$4S_2\dot{f}_l\ddot{f}_l = \frac{1}{W^2}\Big[\dddot{f}_l\sum_{\substack{1\le j\le n\\ j\ne l}}\ddot{f}_j\Big(1+\sum_{\substack{1\le k\le n\\ k\ne l,j}}\dot{f}_k^2\Big)+2\dot{f}_l\ddot{f}_l\sum_{\substack{1\le i<j\le n\\ i,j\ne l}}\ddot{f}_i\ddot{f}_j\Big]. \tag{4.11}$$

Now, taking the derivative of equation (4.11) with respect to $x_s\,(s \neq l)$ we get

$$\begin{aligned} 0 = & -\frac{4\dot{f}_s\ddot{f}_s}{W^4}\Big[\dddot{f}_l\sum_{\substack{1\le j\le n\\ j\ne l}}\ddot{f}_j\Big(1+\sum_{\substack{1\le k\le n\\ k\ne l,j}}\dot{f}_k^2\Big)+2\dot{f}_l\ddot{f}_l\sum_{\substack{1\le i<j\le n\\ i,j\ne l}}\ddot{f}_i\ddot{f}_j\Big] \\ & + \frac{1}{W^2}\Big[\dddot{f}_l\dddot{f}_s\Big(1+\sum_{\substack{1\le k\le n\\ k\ne l,s}}\dot{f}_k^2\Big)+2\dddot{f}_l\dot{f}_s\ddot{f}_s\sum_{\substack{1\le j\le n\\ j\ne l,s}}\ddot{f}_j+2\dot{f}_l\ddot{f}_l\dddot{f}_s\sum_{\substack{1\le j\le n\\ j\ne l,s}}\ddot{f}_j\Big]. \end{aligned}$$

Hence, we conclude that,

$$8\dot{f}_l\ddot{f}_l\dot{f}_s\ddot{f}_s S_2 = \dddot{f}_l\dddot{f}_s\Big(1+\sum_{\substack{1\le k\le n\\ k\ne l,s}} \dot{f}_k^2\Big) + 2(\dddot{f}_l\dot{f}_s\ddot{f}_s + \dot{f}_l\ddot{f}_l\dddot{f}_s)\sum_{\substack{1\le j\le n\\ j\ne l,s}} \ddot{f}_j. \quad (4.12)$$

Finally, derivative of equality (4.12) with respect to x_t ($t \neq l$ and $t \neq s$) gives

$$\dddot{f}_l\dddot{f}_s\dot{f}_t\ddot{f}_t + \dddot{f}_l\dddot{f}_t\dot{f}_s\ddot{f}_s + \dddot{f}_s\dddot{f}_t\dot{f}_l\ddot{f}_l = 0. \quad (4.13)$$

Suppose that $\ddot{f}_l\ddot{f}_s\ddot{f}_t \neq 0$ and $\dddot{f}_l = 0$, then from (4.13) we have that $\dddot{f}_s = 0$ (or $\dddot{f}_t = 0$) and from (4.12) it follows that $8\dot{f}_l\ddot{f}_l\dot{f}_s\ddot{f}_s S_2 = 0$, generating a contradiction. Therefore, $\dddot{f}_l \neq 0$. Similarly, $\dddot{f}_s \neq 0$ and $\dddot{f}_t \neq 0$. Then, from (4.13) we have

$$\frac{\dddot{f}_l}{\dot{f}_l\ddot{f}_l}\frac{\dddot{f}_s}{\dot{f}_s\ddot{f}_s} + \frac{\dddot{f}_l}{\dot{f}_l\ddot{f}_l}\frac{\dddot{f}_t}{\dot{f}_t\ddot{f}_t} + \frac{\dddot{f}_s}{\dot{f}_s\ddot{f}_s}\frac{\dddot{f}_t}{\dot{f}_t\ddot{f}_t} = 0.$$

We conclude that there is a constant nonzero α_l such that $\dddot{f}_l = \alpha_l\dot{f}_l\ddot{f}_l$. Substituting in (4.11) we obtain

$$4S_2W^2 = \alpha_l\sum_{\substack{1\le j\le n\\ j\ne l}} \ddot{f}_j\Big(1+\sum_{\substack{1\le k\le n\\ k\ne l,j}} \dot{f}_k^2\Big) + 2\sum_{\substack{1\le i<j\le n\\ i,j\ne l}} \ddot{f}_i\ddot{f}_j.$$

Differentiating this identity with respect to variable x_l, we get $\dot{f}_l\ddot{f}_l S_2 = 0$, contradicting the fact $\ddot{f}_l\ddot{f}_s\ddot{f}_t \neq 0$. Thus, we must have $\ddot{f}_l\ddot{f}_s\ddot{f}_t = 0$. Hence we conclude that,

$$\begin{aligned}\sigma_3(\ddot{f}_1,\ldots,\ddot{f}_n) &= 0\\ \sigma_4(\ddot{f}_1,\ldots,\ddot{f}_n) &= 0.\end{aligned}$$

Implying that at least $n-2$ derivatives $\ddot{f}_l$ vanish. However, since $S_2 \neq 0$, from the equation (4.1), it follows that it is not possible to have more than $n-2$ vanishing second derivatives, $\ddot{f}_l = 0$. Hence, we can assume that $\ddot{f}_1 = \cdots = \ddot{f}_{n-2} = 0$, $\ddot{f}_{n-1} \neq 0$ and $\ddot{f}_n \neq 0$, and consequently expression (4.1) becomes

$$0 \neq W^4 S_2 = \ddot{f}_{n-1}\ddot{f}_n \alpha$$

for some real constant α. Therefore,

$$\begin{aligned} 0 &\neq 4W^2 \dot{f}_{n-1}\ddot{f}_{n-1} S_2 = \dddot{f}_{n-1}\ddot{f}_n \alpha \\ 0 &\neq 8\dot{f}_{n-1}\ddot{f}_{n-1}\dot{f}_n\ddot{f}_n S_2 = \dddot{f}_{n-1}\dddot{f}_n \alpha \end{aligned}$$

hence we conclude that there exists a constant β such that $\dddot{f}_{n-1} = \beta \dot{f}_{n-1}\ddot{f}_{n-1} \neq 0$ and, then, $4W^2 S_2 = \ddot{f}_n \alpha\beta$. Again, the derivative with respect to x_{n-1} implies that

$$8\dot{f}_{n-1}\ddot{f}_{n-1} S_2 = 0,$$

contradicting the fact $\ddot{f}_{n-1} \neq 0$. This concludes the proof of the theorem. $\square$

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The conservative principle for differential forms on manifolds with boundary

Levi Lopes de Lima[†]

To my dear friend Barnabé Pessoa Lima, on the occasion of his 60th birthday.

Abstract: In this short note we introduce a notion of conservativeness for the heat semigroup associated to the Hodge Laplacian acting on absolute differential forms on a noncompact manifold with a (possibily noncompact) boundary. Assuming that reflected Brownian motion on the underlying manifold is conservative and imposing suitable lower bounds on the Weitzenböck curvature operator and on the shape operator of the boundary we then show that the corresponding conservative principle holds. This extends to our setting a previous result by Masamune [M] in the boundaryless case. A key ingredient in the proof is a domination property for the heat semigroup which follows from a Feynman-Kac formula recently proved by the author [dL1].

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1 Preliminary notions and statement of the main result

Throughout this note we consider a noncompact Riemannian manifold (X,g) of dimension $n \geq 2$. We assume that X carries a (possibly noncompact) boundary Σ, which is oriented by the inward unit normal vector ν. Also, we assume that X is "geodesically complete" in the sense that any geodesic avoiding Σ is defined for all positive time. For each $0 \leq p \leq n$ we denote by $\mathcal{A}^p(X)$ the space of smooth differential p-form on X and by $(\,,\,)_p$ the L^2 inner product on $\mathcal{A}^p(X)$ induced by $g = \langle\,,\,\rangle$. Finally, we represent by $B = -\nabla\nu$ the shape operator of Σ.

Let X_t^x be reflected Brownian motion starting at $x \in X$ [IW, Hs2, dL1]. This is a continuous stochastic process driven by $-\frac{1}{2}\Delta_0$, where Δ_0 is the (nonnegative) Laplacian acting on bounded functions satisfying Neumann boundary condition along Σ[1].

In general, X_t^x might fail to be a Markov process. More precisely, let $\widehat{X} = X \cup \{\infty\}$ be the one-point compactification of X and define

$$\mathbf{e}(x) = \inf\{t \geq 0; \mathsf{X}_t^x = \infty\}.$$

For obvious reasons, $\mathbf{e}$ is called the extinction time of X_t^x. Now, the Markov property for X_t^x might not hold precisely because the process might be explosive in the sense that $\mathbf{e} \not\equiv +\infty$.

This somewhat annoying explosiveness property can be reformulated in analytical terms as follows. It is not hard to check that the semigroup generated by $-\frac{1}{2}\Delta_0$ is given by

[1]Thus, our sign convention is so that $\Delta_0 = -d^2/dx^2$ on $\mathbb{R}$.

$$(e^{-\frac{1}{2}t\Delta_0}f)(x) = \mathbb{E}(f(\mathsf{X}_t^x)\chi_{\{t<\mathbf{e}(x)\}}), \tag{1.1}$$

where f is a bounded, smooth function on X satisfying Neumann boundary condition and χ_E is the indicator function of a subset E. It follows that $t \mapsto e^{-\frac{1}{2}t\Delta_0}$ is a (local) positive preserving, contraction semigroup on the space of such functions. Applying (1.1) with $f = \mathbf{1}$, the function identically equal to 1, we get

$$(e^{-\frac{1}{2}t\Delta_0}\mathbf{1})(x) = \mathbb{P}[t < \mathbf{e}(x)].$$

So in general we have $e^{-\frac{1}{2}t\Delta_0}\mathbf{1} \leq \mathbf{1}$ but being explosive means precisely that $e^{-\frac{1}{2}t\Delta_0}\mathbf{1} \not\equiv \mathbf{1}$ for some (and hence any) $t > 0$. This means that constant functions are *not* preserved by the semigroup.

Another way of expressing this sub-Markov property of X_t relies on the well-known fact that the associated semigroup can be represented by convolution against a smooth kernel; see [B, Chapter 4]. More precisely,

$$(e^{-\frac{1}{2}t\Delta_0}f)(x) = \int_X K_0(t;x,y)f(y)dX_y,$$

where K_0 is the Neumann heat kernel, that is, the fundamental solution of the initial value problem associated to the heat operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\Delta_0$$

with Neumann boundary condition along Σ. Thus, in general we have

$$\int_X K_0(t;x,y)dX_y \leq 1,$$

but again in the explosive case the strict equality holds for some $t > 0$. In this case we are not allowed to interpret K_0 as a transition probability density function for X_t.

The following proposition summarizes the discussion above.

Proposition 1.1. *The following are equivalent:*

1. X_t^x *is non-explosive in the sense that* $\mathbf{e} \equiv +\infty$*;*
2. *For some/any* $t > 0$ *and any* $x \in X$, $K_0(t; x, \cdot)$ *is a probability density function on* X.
3. *For some/any* $t > 0$, $e^{-\frac{1}{2}t\Delta_0}\mathbf{1} = \mathbf{1}$*;*
4. *For some/any* $t > 0$, $(e^{-\frac{1}{2}t\Delta_0} f, \mathbf{1})_0 = (f, \mathbf{1})_0$, *for any compactly supported* $f \in \mathcal{A}^0(X)$ *satisfying Neumann boundary condition;*

We now introduce the following terminology.

Definition 1.1. *If any of the conditions in Proposition 1.1 happens then we say that the conservative principle holds on* $\mathcal{A}^0(X)$.

This property means that the desired probabilistic interpretation for K_0 is restored so that X_t is turned into a genuine Markov process. Equivalently, constant functions are preserved by the semigroup. As we will see later on, this property admits a natural extension to differential forms; see Definition 1.3 below.

It is not hard to exhibit examples of noncompact, geodesically complete manifolds for which the conservative principle on $\mathcal{A}^0$ does not hold; see for instance [A] for the boundaryless case. The following general result due to Gregor'yan provides a sufficient condition for conservativeness.

Theorem 1.1. *Let* (X, g) *be as above and assume that there exists* $x_0 \in X$ *such that*

$$\int^{+\infty} \frac{r dr}{\log \operatorname{vol}(B_r(x_0))} = +\infty .$$

Then the conservative principle holds on $\mathcal{A}^0(X)$. *In particular,* *this is the case if* $\mathrm{vol}(B_r(x_0)) \le Ce^{r^2}$.

Proof. Either adapt the argument in the proof of [G, Theorem 9.1] or appeal to the general abstract result in [St, Theorem 4]. $\square$

So far this is the best available sufficiency result for conservativeness on $\mathcal{A}^0$. Here we will be interested in generalizations to differential forms of a slightly weaker criterium which involves imposing curvature conditions both in the interior and along the boundary. In the boundaryless case this is due to Yau [Y].

Theorem 1.2. *If* (X,g) *is as above, assume that both the Ricci tensor and the shape operator are uniformly bounded from below. Then the conservative principle holds on* $\mathcal{A}^0(X)$ *.*

Proof. See Remark 3 for a simple proof based on the Feynman-Kac formula for differential 1-forms proved in [Hs1, dL1]. $\square$

In order to state our main result, we need to describe the analogue of $e^{-\frac{1}{2}t\Delta_0}$ acting on p-forms. Let d be the exterior differential acting on forms and $d^\star = \pm \star d \star$ be the codifferential, where $\star$ is the Hodge star operator. In fact, as an operator on p-forms, $d^\star = (-1)^{np+n+1} \star d \star$. Recall that the pointwise inner product of differential forms, denoted $\langle \alpha, \beta \rangle$, is the Hodge dual to $\alpha \wedge \star\eta$. Since

$$d\alpha \wedge \star\beta = \alpha \wedge \star d^\star \beta + d(\alpha \wedge \star\beta), \quad \alpha \in \mathcal{A}^{p-1}(X), \quad \beta \in \mathcal{A}^p(X),$$

if we assume further that $\alpha \wedge \star\beta$ is compactly supported, Stokes theorem gives

$$\int_X \langle d\alpha, \beta \rangle \, dX = \int_X \langle \alpha, d^\star \beta \rangle \, dX + \int_\Sigma \alpha \wedge \star \beta.$$

Given $\alpha \in \mathcal{A}^p(X)$, its restriction to Σ decomposes into its tangential and normal components, namely,

$$\alpha = \alpha_{\mathrm{t}} + \alpha_{\mathrm{n}}.$$

We note that the star operator intertwines the corresponding orthogonal projections. We then have

$$(\alpha \wedge \star\beta)_{\mathrm{t}} = \alpha_{\mathrm{t}} \wedge (\star\beta)_t = \alpha_{\mathrm{t}} \wedge \star\beta_{\mathrm{n}},$$

so we obtain *Green's formula*:

$$\int_M \langle d\alpha, \beta \rangle \, dX = \int_X \langle \alpha, d^\star \beta \rangle \, dX + \int_\Sigma \alpha_{\mathrm{t}} \wedge \star \beta_{\mathrm{n}}.$$

From this we have

$$\begin{aligned} \int_X \langle \Delta_p \alpha, \beta \rangle \, dX &= \int_X (\langle d\alpha, d\beta \rangle + \langle d^\star \alpha, d^\star \beta \rangle) \, dX \\ &\quad + \int_\Sigma ((d^\star \alpha)_{\mathrm{t}} \wedge \star\beta_{\mathrm{n}} - \beta_{\mathrm{t}} \wedge \star (d\alpha)_{\mathrm{n}}), \end{aligned} \qquad (1.2)$$

where the Hodge Laplacian acting on p-forms is given by

$$\Delta_p = (d + d^\star)^2 = dd^\star + d^\star d.$$

Definition 1.2. *We say that a p-form α is* absolute *if $\alpha_{\mathrm{n}} = 0$ and $(d\alpha)_{\mathrm{n}} = 0$.*

It follows from (1.2) that under this boundary condition, Δ_p is formally selfajoint because the boundary integral vanishes and we get

$$\int_X \langle \Delta_p \alpha, \beta \rangle \, dX = \int_X (\langle d\alpha, d\beta \rangle + \langle d^\star \alpha, d^\star \beta \rangle) \, dX. \tag{1.3}$$

Moreover, absolute boundary conditions are of elliptic type so we may apply elliptic theory and spectral theory to define the corresponding heat semigroup $e^{-\frac{1}{2}t\Delta_p^{\mathrm{a}}} : L^2\mathcal{A}^p(X) \to L^2\mathcal{A}^p(X)$ [Sc1, Sc2]. Thus, if $A_{\mathrm{a}}^p(X)$ is the space of absolute p-forms and $\omega_0 \in L^2\mathcal{A}_{\mathrm{a}}^p(X) \cap L_{\mathrm{a}}^\infty \mathcal{A}^p(X)$ then $\omega_t = e^{-\frac{1}{2}t\Delta_p}\omega_0 \in L^2\mathcal{A}_{\mathrm{a}}^p(X)$ for any $t > 0$ and moreover it solves the heat equation

$$\frac{\partial \omega_t}{\partial t} + \frac{1}{2}\Delta_p \omega_t = 0, \quad \lim_{t \to 0} \omega_t = \omega_0. \tag{1.4}$$

Also,

$$\lim_{t \to +\infty} e^{-\frac{1}{2}t\Delta_p} = \Pi_{\mathcal{H}_{\mathrm{a}}^p(X)}, \tag{1.5}$$

the orthogonal projection onto the space $\mathcal{H}_{\mathrm{a}}^p(X)$ of absolute L^2 harmonic p-forms. Note that from (1.3) we see that $\omega \in \mathcal{H}_{\mathrm{a}}^p(X)$ if and only if $d\omega = 0$ and $d^\star \omega = 0$. In particular, any $f \in \mathcal{H}_{\mathrm{a}}^0(X)$ is constant.

Inspired by [V, M] we now extend the notion of conservativeness for p-forms on a noncompact manifold with boundary.

Definition 1.3. *We say that the conservative principle holds on $\mathcal{A}^p(X)$ if the equality*

$$\left(e^{-\frac{1}{2}t\Delta_p}\omega, \eta\right)_p = (\omega, \eta)_p \tag{1.6}$$

holds for any compactly supported $\omega \in \mathcal{A}_a^p(X)$ and any bounded $\eta \in \mathcal{H}_{\mathrm{a}}^p(X)$.

This means that absolute, bounded L^2 harmonic p-forms are preserved by the heat semigroup $e^{-\frac{1}{2}t\Delta_p}$.

Our main result provides a criterium for the validity of this principle in terms of certain lower bounds on the curvature. The first notion of curvature we use has to do with the so-called *Weitzenböck decomposition* on p-forms, namely,

$$\Delta_p = \Box_p + R_p,$$

where $\Box_p$ is the Bochner Laplacian and R_p is the *Weitzenböck operator*, a (pointwise) selfadjoint operator acting on p-forms whose local expression depends on the curvature tensor of (X,g) [Ro]. We note that $R_1 = \mathrm{Ric}$ and since $\star R_p = R_{n-p}\star$, this also determines R_{n-1}. However, the structure of R_p, $2 \le p \le n-2$, is notoriously hard to grasp.

The other curvature operator appearing in our main result is obtained by first extending the shape operator $B = -\nabla_\nu$ of Σ to $TM|_\Sigma$ by declaring that $B\nu = 0$ and then extending this to act on p-forms restricted to the boundary as the selfadjont operator $\mathcal{B}_p$ given by

$$(\mathcal{B}_p\omega)(e_1,\cdots,e_p) = \sum_i \omega(e_1,\cdots,Be_i,\cdots,e_p).$$

Notice that $(\mathcal{B}_p\omega)_{\mathrm{n}} = 0$ for any ω, that is, $\mathcal{B}_p\omega$ is always tangential. To determine the eigenvalues of $\mathcal{B}_p$ restricted to tangential forms we choose e_i so that $Be_i = \kappa_i e_i$, where κ_i are the principal curvatures of Σ. It is then immediate to check that

$$(\mathcal{B}_p\omega)(e_{i_1},\cdots,e_{i_p}) = \left(\sum_j \kappa_{i_j}\right)\omega(e_{i_1},\cdots,e_{i_p}),$$

which shows that the sums in the brackets are the (possibly nonzero) eigenvalues of $\mathcal{B}_p$.

To these curvature invariants we attach the functions

$$r_{(p)} : X \mapsto \mathbb{R}, \quad r_{(p)}(x) = \inf_{|\omega|=1} \langle R_p(x)\omega, \omega \rangle,$$

and

$$\kappa_{(p)} : \Sigma \mapsto \mathbb{R}, \quad \kappa_{(p)}(x) = \inf_{1 \le i_1 < \cdots < i_p \le n-1} \kappa_{i_1}(x) + \cdots + \kappa_{i_p}(x),$$

which record the corresponding least eigenvalues. With this terminology at hand we can finally state our main result.

Theorem 1.3. *Let (X, g) be as above and assume that the conservative principle holds for $\mathcal{A}^0(X)$. Assume also that for some $1 \le p \le n-1$ we have $r_{(p)} \ge c_1$ and $\kappa_{(p)} \ge c_2$ for some $c_1, c_2 > -\infty$. Then the conservative principle holds on $\mathcal{A}^p(X)$.*

Corollary 1.1. *Let (X, g) be as above and assume that both the Ricci tensor and the shape operator are uniformly bounded from below. Then the conservative principle holds on $\mathcal{A}^1(X)$.*

Proof. Just combine Theorem 1.3 with Theorem 1.2. □

The proof of Theorem 1.3 is presented in Section 3 below. It crucially uses a semigroup domination property for the heat semigroup $e^{-\frac{1}{2}t\Delta_p}$ which is a consequence of the Feynman-Kac formula proved in [dL1]. This is explained in the next section. We also briefly discuss a version of Theorem 1.3 for spinors in Section 4; see Theorem 4.1. In fact, these results can be extended to a much larger class of Laplace-type operators acting on sections of a vector bundle over X which satisfy suitable boundary conditions. We hope to address these questions in this generality in a forthcoming paper [dL2].

2 The Feynman-Kac formula and semigroup domination

Here we show how the Feynman-Kac formula proved in [dL1] leads to a semigroup domination result for $e^{-\frac{1}{2}t\Delta_p}$ under suitable lower bound assumptions on R_p and $\mathcal{B}_p$; see also [Hs2] for an important previous contribution.

We start with a discussion of the Feynman-Kac formula. As usual, we resort to the so-called Eells-Elworthy-Malliavin approach to stochastic analysis on manifolds, as exposed in [Hs1]. The first step is to rephrase Definition 1.2 in terms of the natural orthogonal projections $\Pi_{\rm t}$ and $\Pi_{\rm n}$ whose ranges are formed by tangential and normal components of a form restricted to Σ, respectively.

Proposition 2.1. *[dL1, Proposition 5.1] A differential p-form ω is absolute if and only if*

$$\Pi_{\rm n}\omega = 0, \quad \Pi_{\rm t}(\nabla_\nu - \mathcal{B}_p)\omega = 0. \tag{2.1}$$

Thus, absolute boundary conditions are of mixed type in the intermediate range $1 \leq p \leq n-1$, that is, they are Dirichlet in normal directions and Robin in tangential directions.

Let X_t, $t \geq 0$, be the (normally) reflected Brownian motion on X starting at some x_0. We assume that the conservative principle holds on $\mathcal{A}^0$, since this is required in Theorem 1.3. In view of Proposition 1.1, this means that X_t is non-explosive, so the sample paths X_t^x remain in X for all time.

Recall that $\mathsf{X}_t = \pi\widetilde{\mathsf{X}}_t$, where $\pi : P_{\rm SO}(X) \to X$ is the principal bundle of oriented orthonormal frames and $\widetilde{\mathsf{X}}_t$ is the *horizontal* reflected Brownian motion starting at some $\widetilde{x}_0 \in \pi^{-1}(x_0)$, whose anti-development is the standard Brownian motion b_t in $\mathbb{R}^n$. Formally, $\widetilde{\mathsf{X}}_t$ satisfies the stochastic differential equation

$$d\widetilde{\mathsf{X}}_t = \sum_{i=1}^{n} H_i(\widetilde{\mathsf{X}}_t) \circ db_t^i + \nu^\dagger(\widetilde{\mathsf{X}}_t) d\lambda_t, \tag{2.2}$$

where $\{H_i\}_{i=1}^n$ are the fundamental horizontal vector fields, the dagger means the standard equivariant lift (scalarization) of tensor fields on X to $P_{\mathrm{SO}}(X)$ and λ_t is the boundary local time associated to X_t.

We now consider $M_{\varepsilon,t} \in \mathrm{End}(\wedge^p \mathbb{R}^n)$ satisfying

$$dM_{\varepsilon,t} + M_{\varepsilon,t}\left(\frac{1}{2}R_p^\dagger(\widetilde{\mathsf{X}}_t)dt + \mathcal{B}_{p,\varepsilon}^\dagger(\widetilde{\mathsf{X}}_t)d\lambda_t\right) = 0, \quad M_{\varepsilon,0} = I, \tag{2.3}$$

where $\varepsilon > 0$ and

$$\mathcal{B}_{p,\varepsilon}^\dagger = \mathcal{B}_p^\dagger + \varepsilon^{-1}\Pi_{\mathrm{n}}^\dagger. \tag{2.4}$$

It is known that, as $\varepsilon \to 0$, $M_{\varepsilon,t}$ converges in L^2 to an adapted, right-continuous multiplicative functional M_t with left limits. Moreover, for all $\varepsilon > 0$ small enough we have the key estimate

$$|M_{\varepsilon,t}| \le \exp\left(-\frac{1}{2}\int_0^t r_{(p)}(\mathsf{X}_s)ds - \int_0^t \kappa_{(p)}(\mathsf{X}_s)d\lambda_s\right), \quad t > 0. \tag{2.5}$$

Now, as before let $\omega_0 \in \mathcal{A}_{\mathrm{a}}^p(M)$, so that $\omega_t = e^{-\frac{1}{2}t\Delta_p}\omega_0 \in \mathcal{A}_{\mathrm{a}}^p(M)$ is the solution to (1.4). Then a simple application of Itô's formula to the process $M_{\varepsilon,t}\omega_{T-t}^\dagger(\widetilde{\mathsf{X}}_t)$, $0 \le t \le T$, yields in the limit $\varepsilon \to 0$ the following fundamental Feynman-Kac formula.

Theorem 2.1. *[dL1, Theorem 5.2] Under the conditions above,*

$$\omega_t^\dagger(\widetilde{x}_0) = \mathbb{E}_{\widetilde{x}_0}(M_t \omega_0^\dagger(\widetilde{\mathsf{X}}_t)). \tag{2.6}$$

Equivalently,

$$\omega_t(x_0) = \mathbb{E}_{x_0}(M_t V_t \omega_0(\mathsf{X}_t)), \tag{2.7}$$

where V_t is the (reversed) stochastic parallel transport acting on differential forms.

An important consequence of this result is the following semigroup domination property for $e^{-\frac{1}{2}t\Delta_p}$.

Theorem 2.2. *Under the conditions above, assume that $R_p \geq c_1$ and $\mathcal{B}_p \geq c_2$, where $c_1, c_2 > -\infty$. Then there exist $C_1 > 0$ and $C_2 > -\infty$ such that*

$$\left| e^{-\frac{1}{2}t\Delta_p} \right| \leq C_1 e^{-C_2 t}, \quad t > 0. \tag{2.8}$$

Proof. From (2.5) we have

$$|M_t| \leq \exp\left(-\left(\frac{1}{2}c_1 + c_2\right)\right)t\right).$$

Since V_t is an isometry, the result follows from (2.7). □

3 The proof of Theorem 1.3

In this section we present the proof of Theorem 1.3. We start with an useful integral identity.

Proposition 3.1. *Let ω and ξ be compactly supported absolute p-forms on X. Then, for any $t > 0$,*

$$\left(e^{-\frac{1}{2}t\Delta_p}\omega - \omega, \xi\right)_p = -\frac{1}{2}\int_0^t \int_X \langle e^{-\frac{1}{2}\tau\Delta_p}\omega, \Delta_p \xi\rangle dX d\tau. \tag{3.1}$$

Proof. We compute:

$$\begin{aligned}
\left(e^{-\frac{1}{2}t\Delta_p}\omega-\omega,\xi\right)_p &= \int_X \langle e^{-\frac{1}{2}t\Delta_p}\omega-e^{-\frac{1}{2}0\Delta_p},\xi\rangle dX \\
&= \int_0^t\int_X \langle \partial_\tau e^{-\frac{1}{2}\tau\Delta_p}\omega,\xi\rangle dX d\tau \\
&\overset{(1.4)}{=} -\frac{1}{2}\int_0^t\int_X \langle \Delta_p e^{-\frac{1}{2}\tau\Delta_p}\omega,\xi\rangle dX d\tau \\
&\overset{(1.3)}{=} -\frac{1}{2}\int_0^t\int_X \langle e^{-\frac{1}{2}\tau\Delta_p}\omega,\Delta_p\xi\rangle dX d\tau,
\end{aligned}$$

as desired. □

We now take a sequence of smooth, compactly supported functions ϕ_i on X such that $0\le\phi_i\le\phi_{i+1}\le 1$, $\phi_i\to\mathbf{1}$ as $i\to+\infty$ and $\partial\phi_i/\partial\nu=0$ along Σ.

Proposition 3.2. *If the conservative principle holds on $\mathcal{A}^0(X)$ then*

$$\zeta_i=\int_0^{+\infty}e^{-t}\int_X K_0(t;\cdot,y)\phi_i(y)dX_y dt$$

is smooth and satisfies: a) $\zeta_i\to 1$; b) $\frac{1}{2}\Delta_0\zeta_i=\phi_i-\zeta_i\to 0$; and c) $\partial\zeta_i/\partial\nu=0$ along Σ.

Proof. First, we have

$$\zeta_i(x)-1=\int_0^{+\infty}e^{-t}\int_X K_0(t;x,y)\left(\phi_i(y)-1\right)dX_y dt,$$

from which a) follows easily. Also,

$$\begin{aligned}
\frac{1}{2}\Delta_0\zeta_i(x) &= \int_0^{+\infty}e^{-t}\int_X\frac{1}{2}\Delta_0 K_0(t;x,y)\phi_i(y)dX_y dt \\
&= -\int_X\left(\int_0^{+\infty}e^{-t}\frac{\partial}{\partial t}K_0(t;x,y)dt\right)\phi_i(y)dX_y \\
&= -\int_X\left(-K_0(0;x,y)+\int_0^{+\infty}e^{-t}K_0(t;x,y)dt\right)\phi_i(y)dX_y,
\end{aligned}$$

which yields b). Finally, c) follows from the fact that the same property holds for K_0. $\square$

Now apply Proposition 3.1 with ω as in Definition 1.3 and $\xi = \zeta_i\eta$, where η is as in Definition 1.3. Since $\Delta_p\eta_i = (\Delta_0\zeta_i)\eta$, we get for each $t > 0$,

$$\begin{aligned}\left|\left(e^{-\frac{1}{2}t\Delta_p}\omega - \omega, \zeta_i\eta\right)_p\right| &\leq \frac{1}{2}\|\Delta_0\zeta_i\|_{L^\infty}\|\eta\|_{L^\infty}\int_0^t \|e^{-\frac{1}{2}\tau\Delta_p}\omega\|_{L^1}d\tau \\ &\leq \frac{C_1}{2}\|\Delta_0\zeta_i\|_{L^\infty}\|\eta\|_{L^\infty}\|\omega\|_{L^1}\int_0^t e^{-C_2\tau}d\tau,\end{aligned}$$

where we used (2.8). By sending $i \to +\infty$, Proposition 3.2 guarantees that the righthand side goes to 0 and that $\zeta_i\eta \to \eta$, so we obtain (1.6), which completes the proof of Theorem 1.3.

Remark 3. *A simpler variant of this argument, which dispenses with Proposition 3.2, yields a proof of Theorem 1.2. We first note that by geodesic completeness we may assume that $\|d\phi_i\|_{L^\infty} \to 0$. Thus, using (3.1) with $\omega = f$ as in item (4) of Proposition 1.1 and $\xi = \phi_i$ we have*

$$\begin{aligned}\left(e^{-\frac{1}{2}t\Delta_0}f - f, \phi_i\right)_0 &= -\frac{1}{2}\int_0^t\int_X \langle e^{-\frac{1}{2}\tau\Delta_0}f, \Delta_0\phi_i\rangle dX d\tau \\ &= -\frac{1}{2}\int_0^t\int_X \langle e^{-\frac{1}{2}\tau\Delta_0}f, d^*d\phi_i\rangle dX d\tau \\ &= -\frac{1}{2}\int_0^t\int_X \langle de^{-\frac{1}{2}\tau\Delta_0}f, d\phi_i\rangle dX d\tau \\ &= -\frac{1}{2}\int_0^t\int_X \langle e^{-\frac{1}{2}\tau\Delta_1}df, d\phi_i\rangle dX d\tau,\end{aligned}$$

where here we assume that $t < \mathbf{e}$, the extinction time of X_t. It follows that

$$\begin{aligned}\left|\left(e^{-\frac{1}{2}t\Delta_0}f-f,\phi_i\right)_0\right| &\leq \frac{1}{2}\|d\phi_i\|_{L^\infty}\int_0^t \|e^{-\frac{1}{2}\tau\Delta_1}df\|_{L^1}d\tau \\ &\leq \frac{C_1}{2}\|d\phi_i\|_{L^\infty}\|df\|_{L^1}\int_0^t e^{-\frac{1}{2}C_2\tau}d\tau, \quad (3.2)\end{aligned}$$

where we used Theorem 2.2 with $p=1$*. By sending* $i\to+\infty$ *we then recover item (4) in Proposition 1.1 for some* $t>0$*, which proves Theorem 1.2.*

Remark 4. *Assume that* $c_1,c_2>0$ *so that* $C_2>0$ *in Theorem 2.2 (with* $p=1$*). By sending* $t\to+\infty$ *(3.2) and using (1.5) we get*

$$\left|\left(\Pi_{\mathcal{H}^0_{\mathrm{a}}}f-f,\phi_i\right)_0\right| \leq \frac{C_1}{2}\|d\phi_i\|_{L^\infty}\|df\|_{L^1}\int_0^{+\infty} e^{-\frac{1}{2}C_2 t}d\tau.$$

so by taking $i\to+\infty$ *we end up with*

$$\left(\Pi_{\mathcal{H}^0_{\mathrm{a}}}f-f,\mathbf{1}\right)_0=0.$$

But $\Pi_{\mathcal{H}^0_{\mathrm{a}}}f$ *is harmonic and hence constant. Thus, if* $\mathrm{vol}(X)=+\infty$ *then* $\Pi_{\mathcal{H}^0_{\mathrm{a}}}f=0$*. But this leads to* $\int_X f\,dX=0$ *for any such* f*, a contradiction. Hence, we have seen that if the Ricci tensor and the shape operator are both uniformly bounded from below by a positive constant then* (X,g) *has finite volume. This is the analogue of Bonnet-Myers theorem in this setting.*

4 The spin conservative principle

Assume that X as above is spin and fix a spin structure. In [dL1, Section 5] we proved a Feynman-Kac formula for the semigroup $e^{-\frac{1}{2}tD^2}$ associated to the Dirac Laplacian D^2, where D is the Dirac operator acting on spinors associated to a metric g on X. This formula was established under the assumption that the conservative principle holds on $\mathcal{A}^0$ and by imposing suitable boundary conditions on spinors along Σ, which include the chilarity and the MIT bag boundary conditions. As a consequence, a semigroup domination result for $e^{-\frac{1}{2}tD^2}$ can be derived if we assume further that the scalar curvature s on X and the mean curvature H on Σ are both uniformly bounded from below. Thus, the following result can be proved by using the obvious variant of the argument leading to Theorem 1.3.

Theorem 4.1. *Let (X,g) be spin and assume that the conservative property holds on $\mathcal{A}^0(X)$. Assume also that both s and H are uniformly bounded from below. Then the spin conservative principle holds for (X,g) in the sense that*

$$\left(e^{-\frac{1}{2}tD^2}\psi,\varphi\right)=(\psi,\varphi),$$

where ψ and φ are spinors satisfying any of the boundary conditions mentioned above, ψ is compactly supported, φ is L^2 harmonic and bounded and $(\,,)$ is the standard L^2 pairing for spinors.

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Some results on compact almost Ricci solitons with null Cotton tensor

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In honor of the 60th Birthday of Professor Barnabé Pessoa Lima.

Abstract: The aim of this paper is to prove that a compact almost Ricci soliton with null Cotton tensor is isometric to a standard sphere provided one of the following conditions holds: the second symmetric function associated to the Schouten tensor is constant and positive; two consecutive symmetric functions associated to the Schouten tensor are non null multiple or some symmetric function associated to the Schouten tensor is constant and the Schouten tensor is positive.

1 Introduction

The concept of almost Ricci soliton was introduced by Pigola et al. in [17], where essentially they modified the definition of a Ricci soliton by permitting to the parameter λ to be a variable function. More precisely, we say that a Riemannian manifold (M^n, g) is an almost Ricci soliton if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

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where Ric and $\mathcal{L}$ stand for the Ricci curvature tensor and the Lie derivative, respectively. We shall refer to this equation as the fundamental equation of an almost Ricci soliton (M^n, g, X, λ). We say that an almost Ricci soliton is shrinking, steady or expanding provided $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively, otherwise we say that it is indefinite. When $X = \nabla f$ for some smooth function f on M^n, we say that it is a gradient almost Ricci soliton. In this case identity (1.1) becomes

$$Ric + \nabla^2 f = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ stands for the Hessian of f. Further, an almost Ricci soliton is trivial provided X is a Killing vector field, otherwise it will be called a non-trivial almost Ricci soliton. We point out that when X is a Killing vector field and $n \geq 3$, we have that M is an Einstein manifold since Schur's lemma ensures that λ is constant.

We highlight that Ricci solitons also correspond to self-similar solutions of Hamilton's Ricci flow, for more details about Ricci soliton see e.g. [8]. In this perspective Brozos-Vázquez, García-Río and Valle-Regueiro [6] observed that some proper gradient Ricci almost solitons correspond to self-similar solutions of the Ricci-Bourguignon flow, which is a geometric flow given by

$$\frac{\partial}{\partial t} g(t) = -2(Ric(t) - kR(t)g(t)),$$

where $k \in \mathbb{R}$ and R stands for the scalar curvature. This flow can be seen as an interpolation between the flows of Ricci and Yamabe. For more details on Ricci-Bourguignon flow we recommend [10].

It is important to emphasize that the round sphere does not admit a (nontrivial) Ricci soliton structure. However, Barros and Ribeiro Jr [3] showed an explicit example of an almost Ricci soliton on the standard sphere. For this reason, it is interesting to know if, in the compact case, this is the unique example with non-constant soliton function λ. In this sense, Barros and Ribeiro Jr [3] proved that a compact gradient almost Ricci soliton with constant scalar curvature must be isometric to a standard sphere. Afterward, Barros, Batista and Ribeiro Jr [1] proved that every compact almost Ricci soliton with constant scalar curvature is gradient. In [5], Costa, Brasil and Ribeiro Jr showed that under a suitable integral condition, a 4-dimensional compact almost Ricci soliton is isometric to the standard sphere $\mathbb{S}^4$. While Ghosh [12] was able to prove that if a compact K-contact metric is a gradient almost Ricci soliton, then it is isometric to a unit sphere. We recall that Barros, Batista and Ribeiro Jr [2] proved that under a suitable integral condition a locally conformally flat compact almost Ricci soliton is isometric to a standard sphere $\mathbb{S}^n$. For more details see, for instance, [1], [2], [12], [15] and [19].

When M is a compact manifold the Hodge-de Rham decomposition theorem (see for instance [20]) asserts that X can be decomposed as a sum of a gradient of a function h and a divergence-free vector field Y, i.e.

$$X = \nabla h + Y,$$

where $divY = 0$. From now on we consider h the function given by this decomposition.

Henceforth we denote by M^n, $n \geq 3$, a compact connected oriented manifold without boundary. Now we remember some ba-

sic facts about symmetric functions. Let A be the Schouten tensor and $\sigma_k(A)$ be the symmetric functions associated to A defined as follows
$$\det(I+tA)=\sum_{k=0}^{n}\sigma_k(A)t^k.$$
Since A is symmetric, then $\binom{n}{k}S_k(A)=\sigma_k(A)$ coincides with the k-th elementary symmetric polynomial of the eigenvalues $\lambda_i(A)$ of A, i.e.,

$$\sigma_k(A)=\sigma(\lambda_1(A),\ldots,\lambda_n(A))=\sum_{i_1<\cdots<i_k}\lambda_{i_1}(A)\cdots\lambda_{i_k}(A), \tag{1.3}$$

for more details about symmetric functions see for instance [14]. It should be emphasized that the assumption of constant scalar curvature is equivalent to require that the trace of the Schouten tensor is constant. Indeed, if we denote by A the Schouten tensor, then $trA=\frac{n-2}{2(n-1)}R$, where R stands for the scalar curvature of M. Since $nS_1(A)=trA$, this is in turn equivalent to require that the first symmetric function of A is constant. Inspired by the historical development on the study of compact almost Ricci soliton. In this paper, we investigate which geometric implication has the assumption that the second symmetric function S_2 associated to the Schouten tensor is constant and positive on a compact almost Ricci soliton. More precisely, we have the following result.

Theorem 1.1. *Let (M^n,g,X,λ) be a non-trivial compact oriented almost Ricci soliton such that the Cotton tensor is identically zero. Then, M^n is isometric to a standard sphere $\mathbb{S}^n$ provided that one of the next condition is satisfied:*

1. *$S_2(A)$ is constant and positive.*

2. *$S_k(A)$ is nowhere zero on M and $S_{k+1}(A) = cS_k(A)$, where $c \in \mathbb{R}\backslash\{0\}$, for some $k = 1,\cdots,n-1$.*

3. *$Ric \geq \frac{R}{n}g$, with $R > 0$, and $\int_M S_k(A)\Delta h \geq 0$ for some $2 \leq k \leq n-1$.*

4. *$S_k(A)$ is constant for some $k = 2,\cdots,n-1$, and $A > 0$.*

We highlight that the symmetric functions associated to the Schouten tensor were used by Hu, Li and Simon [14] to study locally conformally flat manifolds. By assuming that the Weyl tensor vanishes, then the conclusion of item 4 in Theorem 1.1 follows directly from Theorem 1 obtained in [14]. In this direction, we point out that item 1 and item 4 of Theorem 1.1 improve Theorem 1 in [14] for compact almost Ricci solitons under the hypothesis of Cotton tensor identically zero.

Recently Catino, Mastrolia and Monticelli [9] obtained an important classification for gradient Ricci soliton admitting a fourth-order vanishing condition on the Weyl tensor. More precisely, they showed that any n-dimensional $n \geq 4$ gradient shrinking Ricci soliton with fourth order divergence-free Weyl tensor $div^4 W = 0$, is either Einstein, or a finite quotient of $N^{n-k} \times \mathbb{R}$, $k > 0$ the product of a Einstein manifold N^{n-k} with the Gaussian shrinking soliton R^k. We remark that

$$div^4(W) = W_{ijkl;ikjl},$$

where the indexes after the comma are the covariant derivatives. We highlight that in general the condition of divergence free of a tensor does not imply that the tensor is null, for example, $\mathbb{CP}^2$ is an Einstein manifold with Fubini-Study metric, hence its Cotton tensor is null, since the Cotton satisfies $\nabla^l W_{ijkl} = \dfrac{n-3}{n-2}C_{ijk}$, we have that the Weyl tensor is harmonic. However,

$\mathbb{CP}^2$ is not locally conformally flat. This example shows that harmonic Weyl tensor does not imply locally conformally flatness. In this direction, we get the following interesting result about the Cotton tensor of a compact oriented manifold without boundary.

Theorem 1.2. *Let (M,g) be a compact oriented manifold without boundary and C its be the Cotton tensor. Suppose that $divC=0$, then $C=0$.*

It is important to emphasizes that the previous Theorem allows us to suppose $divC=0$ instead of C in the Theorem 1.1. This result has an other interesting consequences, , recently Lopes in [18] proved that the CPE conjecture is true provided that $divC=0$, and by Theorem 1.2 this hypothesis implies imediatly that $C=0$, then one can conclude that the CPE conjecture is true by applying Theorem 1.2 in [9].

The paper is organized as follows: in Section 2 we present some basic notations and definitions; subsection 2.2 is devoted to define Newton transformations associated to a symmetric $(0,2)$ tensor and to compute the divergence of such transformations whereas in subsection 2.3 we establish some integral formulae for compact oriented almost Ricci soliton associated to the Schouten operator. In Section 3 we prove our main result as an application of the integral formulae obtained in the previous section.

2 Preliminaries

2.1 Notations

Let (M^n, g) be a smooth, n-dimensional Riemannian manifold with metric g. We denote by $Rm(X,Y)Z$ the Riemann curvature operator defined as follows

$$Rm(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and we also denote by $Ric(X,Y) = tr(Z \to Rm(Z,X)Y)$ the Ricci tensor, and $R = tr(Ric)$ the scalar curvature. We have the well known formula

$$(div Rm)(X,Y,Z) = \nabla_X Ric(Y,Z) - \nabla_Y Ric(X,Z). \tag{2.1}$$

Let $A = Ric - \frac{R}{2(n-1)} g$ denote the Schouten tensor, which is a $(0,2)$ symmetric tensor. The Weyl tensor is given by

$$Rm = W + \frac{1}{n-2}(A \odot g), \tag{2.2}$$

where $\odot$ means the Kulkarni-Nomizu product defined by the following formula

$$(\alpha \odot \beta)_{ijkl} = \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{ik}\beta_{jl} - \alpha_{jl}\beta_{ik}, \tag{2.3}$$

where α, β are $(0,2)$ tensor. Finally we define the Cotton tensor as follows

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}. \tag{2.4}$$

It is well known that

$$\nabla^l W_{ijkl} = \frac{n-3}{n-2} C_{ijk}. \tag{2.5}$$

From identities (2.4) and (2.5) we see that for $n \geq 4$ if the Weyl tensor vanishes, then the Cotton tensor also vanishes. We also see that when $n = 3$ the Weyl tensor always vanishes, but the Cotton tensor does not vanish in general. We say that a manifold has harmonic Weyl tensor provided that $divW = 0$, where div means the divergence of the tensor. By (2.5) we also have that for $n \geq 4$, the Cotton tensor is identically zero, if and only if, the Weyl tensor is harmonic.

We define $(divC)_{jk} = \nabla_i C_{ijk}$ the divergence of the Cotton tensor. We also define the second order divergence-free Weyl tensor condition $div^2(W) = W_{ijkl;ik}$ and the fourth order divergence-free Weyl tensor $div^4(W) = W_{ijkl;ikjl}$.

2.2 Newton transformations

Let T be a symmetric $(0,2)$ tensor and $\sigma_k(T)$ be the symmetric functions associated to T defined as follows

$$\det(I + sT) = \sum_{k=0}^{n} \sigma_k(T) s^k,$$

where $\sigma_0 = 1$. Since T is symmetric, then $\binom{n}{k} S_k(T) = \sigma_k(T)$ coincides with the k-th elementary symmetric polynomial of the eigenvalues $\lambda_i(T)$ of T, i.e.,

$$\sigma_k(T) = \sigma(\lambda_1(T), \dots, \lambda_n(T)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1}(T) \cdots \lambda_{i_k}(T), \ 1 \leq k \leq n. \quad (2.6)$$

Introduce the Newton transformations $P_k(T) : \mathfrak{X}(M) \to \mathfrak{X}(M)$, arising from the operator T, by the following inductive law

$$P_0(T) = I, \ P_k(T) = \binom{n}{k} S_k(T) I - T P_{k-1}(T), \ 1 \leq k \leq n \quad (2.7)$$

or, equivalently,

$$
\begin{aligned}
P_k(T) &= \binom{n}{k} S_k(T)I - \binom{n}{k-1} S_{k-1}(T)T + \cdots + (-1)^{k-1}\binom{n}{1} S_1(T)T^{k-1} \\
&\quad + (-1)^k T^k.
\end{aligned}
$$

Using the Cayley-Hamilton theorem we get $P_n(T) = 0$.

Note that $P_k(T)$ is a self-adjoint operator that commutes with T for any k. Furthermore, if $\{e_1, \ldots, e_n\}$ is an orthonormal frame on T_pM diagonalizing T, then

$$
(P_k(T))_p(e_i) = \mu_{i,k}(T)_p e_i, \tag{2.8}
$$

where

$$
\mu_{i,k}(T) = \sum_{i_1 < \cdots < i_k, i_j \neq i} \lambda_{i_1}(T) \cdots \lambda_{i_k}(T) = \frac{\partial \sigma_{k+1}}{\partial x_i}(\lambda_1(T), \ldots, \lambda_n(T)).
$$

The divergence of $P_k(T)$ is defined as follows

$$
\mathrm{d}P_k(T) = tr(\nabla P_k(T)) = \sum_{i=1}^{n} \nabla_{e_i} P_k(T)(e_i),
$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. Our aim is to compute the divergence of $P_k(T)$. The following definition is important in the sequel. Define the tensor D by

$$
D_{ijk} = \nabla_i T_{jk} - \nabla_j T_{ik}. \tag{2.9}
$$

Note that when T is the Ricci tensor, then by equation (2.1) $D = divRm$, and when T is the Schouten tensor, then D is just the Cotton tensor.

Lemma 2.1. *Let $P_k(T)$ be the Newton transformations associated with T above defined and let $\{e_1,\dots,e_n\}$ be a local orthonormal frame on M. Then, for all $Z \in \mathfrak{X}(M)$, the divergence of $P_k(T)$ are given recursively as*

$$\begin{aligned} &\mathrm{d}P_0(T) = 0 \\ &\langle \mathrm{d}P_k(T), Z\rangle = -\langle T(\mathrm{d}P_{k-1}(T)), Z\rangle - \sum_{i=1}^{n} D(e_i, Z, P_{k-1}(T)e_i), \end{aligned} \tag{2.10}$$

or equivalently

$$\langle \mathrm{d}P_k(T), Z\rangle = \sum_{j=1}^{k}\sum_{i=1}^{n}(-1)^j D(e_i, T^{j-1}Z, P_{k-j}(T)e_i). \tag{2.11}$$

In particular we have the following.

Corollary 2.1. *If $D = 0$, then the Newton transformations are divergence free: $\mathrm{d}P_k(T) = 0$ for each k.*

2.3 Integral Formulae

Lemma 2.2. *Let (M, g, X, λ) be a compact oriented almost Ricci soliton. For each k, the following integral formula holds:*

$$\int_M \langle \mathrm{d}P_k(A), X\rangle dV_g + c_k \int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)S_k(A) - S_{k+1}(A)\Big) dV_g = 0. \tag{2.12}$$

Note that when the Cotton tensor vanishes Corollary 2.1 implies that

$$\int_M \langle \mathrm{d}P_k(A), X\rangle dV_g = 0. \tag{2.13}$$

Therefore, we obtain the next corollary.

Corollary 2.2. *Let (M, g, X, λ) be a compact oriented almost Ricci soliton such that the Cotton tensor vanishes. Then,*

$$\int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)S_k(A) - S_{k+1}(A)\Big)dV_g = 0. \tag{2.14}$$

3 Proof of the Main Results

Remark 5. *Before presenting the proofs of the results, we recall that the symmetric functions satisfy Newton's inequalities:*

$$S_k(A)S_{k+2}(A) \le S_{k+1}^2(A) \text{ for } 0 \le k < n-1, \tag{3.1}$$

which is a generalized Cauchy-Schwarz inequality. Moreover, if equality occurs for $k=0$ or $1 \le k < n$ with $S_{k+2}(A) \ne 0$, then $\lambda_1(A) = \lambda_2(A) = \ldots = \lambda_n(A)$. As an application, provided that $\lambda_k(A) > 0$ for $1 \le k \le n$, we obtain Gårding's inequalities

$$S_1 \ge S_2^{\frac{1}{2}} \ge S_3^{\frac{1}{3}} \ge \cdots \ge S_n^{\frac{1}{n}}. \tag{3.2}$$

Here equality holds, for some $1 \le k < n$, if and only if, $\lambda_1(A) = \lambda_2(A) = \ldots = \lambda_n(A)$. Note that (3.2) *implies that $S_k^{\frac{k+1}{k}} \ge S_{k+1}$ for $1 \le k < n$. For a proof see for instance [13] Theorem 51, p. 52 or Proposition* 1 *in [7].*

3.1 Proof of Theorem 1.1

Proof. In item 1 we suppose that $S_2(A)$ is constant and positive. Thereby, choosing $k=2$ in (2.14) we obtain

$$\int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)S_2(A) - S_3(A)\Big)dV_g = 0. \tag{3.3}$$

Since $S_2(A)$ is constant we deduce

$$\int_M \big(S_2(A)S_1(A) - S_3(A)\big)dV_g = 0. \tag{3.4}$$

On the other hand,

$$S_1^2(A) - S_2(A) \geq 0, \tag{3.5}$$

by Newton's inequality (3.1). Moreover, equality in (3.5) holds, if and only if, $\lambda_1(A) = \cdots = \lambda_n(A)$, which means that A is umbilical (a multiple of g). In this case it is easy to check that

$$A = \frac{(n-2)R}{2n(n-1)} g. \tag{3.6}$$

We know from (3.5) that $S_1^2(A) \geq S_2(A) > 0$, then $S_1(A)$ does not vanish, this means that either $S_1(A) < 0$ or $S_1(A) > 0$. Now we prove that $S_2(A)S_1(A) - S_3(A)$ is positive or negative, according to the sign of $S_1(A)$.

Indeed, from (3.1) we get $S_2^2(A) - S_1(A)S_3(A) \geq 0$. Supposing $S_1(A) > 0$ we obtain

$$S_2(A)S_1(A) - S_3(A) \geq S_2(A)S_1(A) - \frac{S_2^2(A)}{S_1(A)} = \frac{S_2(A)}{S_1(A)}\left(S_1^2(A) - S_2(A)\right) \geq 0. \tag{3.7}$$

Analogously, if $S_1(A) < 0$ then $S_2(A)S_1(A) - S_3(A) \leq 0$. In both cases $S_2(A)S_1(A) - S_3(A)$ has a sign. Using this fact together with equation (3.4), we get $S_2(A)S_1(A) - S_3(A) = 0$, and hence equality in (3.1), obtaining identity (3.6). Therefore (M^n, g) is an Einstein manifold and we are in position to apply Corollary 1 in [1] to conclude that (M^n, g) is isometric to a standard sphere $\mathbb{S}^n$.

Proceeding, in item 2 we assume that $S_k(A)$ is nowhere zero on M and $S_{k+1}(A) = cS_k(A)$, where $c \in \mathbb{R}\backslash\{0\}$, for some $k = 1, \cdots, n-1$. Thus we can use Corollary 2.2 to infer

$$\int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)cS_k(A) - cS_{k+1}(A)\Big)dV_g = 0 \tag{3.8}$$

and

$$\int_M \Big(\big(S_1(A) + \frac{1}{n}\Delta h\big)S_{k+1}(A) - S_{k+2}(A)\Big)dV_g = 0. \tag{3.9}$$

By hypothesis $S_{k+1}(A) = cS_k(A)$, whence using (3.9) and (3.8) we deduce

$$\int_M \big(cS_{k+1}(A) - S_{k+2}(A)\big)dV_g = 0. \tag{3.10}$$

Using once more that $S_{k+1}(A) = cS_k(A)$ we invoke inequality (3.1) to get $S_k(A)(cS_{k+1}(A) - S_{k+2}(A)) \geq 0$. Using a similar argument used in the previous case we conclude that M is Einstein, which finishes the proof. $\square$

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From geometric analysis to classical geometry: 15 years talking about geometry with Professor Pessoa Lima

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Dedicated to my friend, Prof. Barnabé Pessoa Lima on occasion of his 60th birthday.

Abstract: In this note we present some of the joint work develloped with Professor Pessoa Lima. The main themes we worked togehter are related to the geometry of p-laplacian operator on manifolds and translation surfaces.

1 Introduction

When we arrived, in the year of 2002, in Teresina to begin our activities as a Professor at the Universidade Federal do Piauí - UFPI, there was just one doctor in Geometry in the department of Mathematics of that Institution, Professor Barnabé Pessoa Lima. Pessoa Lima's research area, focused on Omori-Yau Maximum Principle and L_r operators and had not apparent ties to the area of my PhD, Global Geometry of Manifolds and Comparison Theorems. I was actually uncomfortable with that situation because it was clear for me the importance of stablish-

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ing and working in partnership, mainly because we were living away from large research centers. We trully searched for contact points in our areas but we didnot achieved many advances. Nevertheless, all these efforts became responsible for the great mutual friendship and admiration we develop.

Shortly after, with several teachers approved in public tender for Departmento of Mathematics of UFPI, the team of researchers in Geometry increased. Between 2007 and 2008 I finished my postdoctoral studies under the supervision of Professor Levi Lopes de Lima, Universidade Federal do Ceará - UFC. It was fruitful year of learning and particularly special year because it brought in its bag new knowledge acquired. In addition to the techniques learned with Prof Lopes de Lima, subjects related to the research themes of Professors José Fábio Bezerra Montenegro and Gregorio Pacelli Bessa. From these themes I suggested to Professor Pessoa Lima that we could work with the p-laplacian operator on manifolds and find results and eigenvalue estimates similar to those related to the laplacian operator. Together with Fábio Montenegro, we obtained a series of interesting results [19], theorems 2.1, 2.2, 2.4 and 2.5. Later, in a partnership with Professor Pessoa Lima, using oscillation techniques [20], we obtained the theorems 2.6 and 2.7. These results will be presented in subsection 2.1. Several questions remained open to problems of eigenvalue with the p-laplacian, for example, to obtain estimates for the first eigenvalue for Steklov's problem and for the Robin problem.

Another important and interesting topic that I had the opportunity to interact with Prof Pessoa Lima was when I was invited to join the team formed by Professors Juscelino Pereira

Silva, Paulo Alexandre Araujo Sousa and Barnabé Pessoa Lima for the study of translational hypersurfaces. Translational surfaces form a rich class of surfaces that allow the elaboration of numerous questions each of which contribute with extremely interesting solutions and examples of hypersurfaces. These results will be presented in subsection 2.2.

2 Statement of results

2.1 The *p*-laplacian operator

The Laplace-Beltrami operator on a riemannian manifold (M,g), its spectral theory and the relations between its first eigenvalue and geometrical data of the manifold, such as curvatures, diameter, injectivity radius, volume, has been extensively studied in the recent mathematical literature. In the last few years, another operator, called p-laplacian, arising from problems of glaceology, non-newtonian fluids, nonlinear elasticity, and well-known in problems of Nonlinear Partial Differential Equations came to the light of Geometry. Since then, geometers showed that this singular operator exhibit some very interesting analogies with the laplacian, for instance [29], [30] or [14].

Let (M,g) be a smooth riemannian manifold and $\Omega \subset M$ a domain. For $1 < p < \infty$, the p-laplacian on Ω is defined by

$$\triangle_p(u) = -\mathrm{div}\Big[\|\nabla u\|^{p-2}(\nabla u)\Big] \tag{2.1}$$

this operator appears naturally from the variational problem associated to the energy functional

$$E_p : \mathcal{W}_0^{1,p}(\Omega) \to \mathbb{R} \qquad \text{given by} \qquad E_p(u) = \int_\Omega \|\nabla u\|^p \, d\Omega$$

where, as usual, $\mathcal{W}_0^{1,p}(\Omega)$ denotes the Sobolev space given by the closure of $\mathcal{C}^\infty(\Omega)$ functions compactly supported in Ω, for the norm

$$\|u\|_{1,p}^p = \int_\Omega |u|^p \, d\Omega + \int_\Omega \|\nabla u\|^p \, d\Omega$$

Observe that, when $p = 2$, $\triangle_2$ is just the Laplace-Beltrami operator. We are interested in the nonlinear eigenvalue problem

$$\triangle_p u + \lambda |u|^{p-2} u = 0 \tag{2.2}$$

Since solutions for this problem, for arbitrary $p \in (1, \infty)$ are only locally $\mathcal{C}^{1,\alpha}$ (exceptions for the case $p = 2$), they must be described in the sense of distribution, that is, $u \in \mathcal{W}_0^{1,p}(\Omega)$, not identically 0 is an eigenfunction, associated to the eigenvalue λ, if

$$\int_\Omega \|\nabla u\|^{p-2} g(\nabla u, \nabla \phi) \, d\Omega = \lambda \int_\Omega |u|^{p-2} u \phi \, d\Omega$$

for any test function $\phi \in \mathcal{C}_0^\infty(\Omega)$.

The spectral set, $\operatorname{spec}_p(M)$, of eigenvalues of (2.2) is an unbounded subset of $[0, \infty)$, as quoted in [29] (see also [23]) whose infimum $\inf \operatorname{spec}_p = \mu_{1,p}(\Omega)$ is an eigenvalue. It is also known (see [29]) that the first eigenvalue is simple and the first eigenfunction radial for geodesic balls on space-forms.

Actually, on Euclidian domains, an argument based on a version of the Ljusternik-Schnirelman principle, An Lê (2005), in [17] shows not only for the Dirichlet nonlinear eigenvalue problem, but also for Neumann and some other interesting classes of non-linear eigenvalue problems associated to the p-laplacian that, for bounded domains, Ω in Euclidian space, $\mathbb{R}^n$ one has:

i. There exists a nondecreasing sequence of nonnegative eigenvalues obtained by the Ljusternik-Schnirelman principle, $(\lambda_n)_n$, tending to ∞ as $n \to \infty$ (though it is not known whether there is or not a non Ljusternik-Schnirelman eigenvalue).

ii. The first eigenvalue λ_1 is simple and only eigenfunctions associated with λ_1 do not change sign.

iii. The set of eigenvalues is closed.

iv. The first eigenvalue λ_1 is isolated.

In An Lê's article, results due to Lindqvist 1995 [23], Anane and Tsouli 1996 [2], and Azorero and Alonso 1987 [1] are naturally extended to a wider class of problems.

It is interesting to point out that an investigation on the variational methods used in [17] can be easily extended to the setup of domains on Riemannian manifolds. More precisely, if Ω is a bounded domain with smooth boundary in a Riemannian manifold then conclusions [i] to [iv] above hold.

Now, let $\Omega \subset M$ a domain. The *p-fundamental tone of* Ω, denoted by $\mu_p^*(\Omega)$ is defined as follows:

$$\mu_p^*(\Omega) = \inf\left\{\frac{\int_\Omega \|\nabla f\|^p \, d\Omega}{\int_\Omega |f|^p \, d\Omega}; f \in \mathcal{W}_0^{1,p},\ f \neq 0\right\}$$

when Ω is a domain with compact closure and nonempty piecewise smooth boundary $\partial\Omega$, then $\mu_p^*(\Omega)$ coincides with the first eigenvalue of the eigenvalue problem with Dirichlet condition, $u|_{\partial\Omega} = 0$, by Rayleigh's theorem.

Observe that if $\Omega_1 \subset \Omega_2$ are bounded domains, then $\mu_p^*(\Omega_1) \geq \mu_p^*(\Omega_2) \geq 0$. Thus one may obtain the p-fundamental tone $\mu_p^*(M)$ of an open riemannian manifold (i.e., complete noncompact) as the limit

$$\mu_p^*(M) = \lim_{r\to\infty} \mu_p^*(B_r(q))$$

where $B_r(q)$ is the geodesic ball of radius r centered at $q \in M$.

When $p = 2$ the p-laplacian is simply the laplacian and the p-fundamental tone is simply called the fundamental tone. Interesting estimates on the fundamental tone for the Laplace-Beltrami operator on a riemannian manifold have been obtained by Gregório P. Bessa and the second author (see, for instance [3] and [4]). This paper presents an attempt to extend their variational argument to the p-laplacian.

Following closely [3] and [4] we introduce a geometric invariant associated to certain spaces of vector fields that will be used to give lower bounds for the fundamental tone for p-laplacian.

Definition 2.1. *Let $\Omega \subset M$ be a domain with compact closure in a smooth riemannian manifold (M^n, g). Let $\mathfrak{X}(\Omega)$ be the set of all smooth vector fields, X, on Ω with sup norm $\|X\|_\infty = \sup_\Omega \|X\| < \infty$ (where $\|X\| = g(X,X)^{1/2}$) and $\inf_\Omega \operatorname{div} X > 0$. Define $c(\Omega)$ by*

$$c(\Omega) := \sup\left\{\frac{\inf_\Omega \operatorname{div} X}{\|X\|_\infty};\ X \in \mathfrak{X}(\Omega)\right\} \tag{2.3}$$

As remarked in [3] $\mathfrak{X}(\Omega)$ is a nonvoid set of smooth vector fields on Ω

With this definition at hand we obtained (in [19]) the following result:

Theorem 2.1. *Let $\Omega \subset M$ be a domain ($\partial\Omega \neq \emptyset$) in a riemannian manifold, M. Then*

$$\mu_p^*(\Omega) \geq \frac{c(\Omega)^p}{p^p} > 0 \tag{2.4}$$

where $c(\Omega)$ is the constant given in (2.3)

To present the second variational estimate we need to introduce some preliminary definitions (see [4]) which will allow us to deal with divergence of vector fields in a weak sense.

Definition 2.2 (Weak divergence)**.** *Let* (M,g) *be a riemannian manifold and* $X \in \mathbf{L}^1_{loc}(M)$ *(in the sense that* $\|X\| \in L^1_{loc}(M)$*) A function* $h \in L^1_{loc}(M)$ *is said to be a weak divergence of* X*, denoted by* $h = \mathrm{D}ivX$ *if for every* $\phi \in \mathcal{C}^\infty_0(M)$ *it holds*

$$\int_M \phi h \, dM = -\int_M g(\nabla\phi, X)\, dM \tag{2.5}$$

We denote by $\mathcal{W}^{1,1}(M)$*, the set of vector fields of* M *possessing weak divergence.*

If $X \in \mathcal{W}^{1,1}(M)$ and $f \in \mathcal{C}^\infty(M)$ then $fX \in \mathcal{W}^{1,1}(M)$ with

$$\mathrm{D}iv(fX) = g(\nabla f, X) + f\mathrm{D}ivX.$$

In particular for $f \in \mathcal{C}^\infty_0(M)$ we have that

$$\int_M \mathrm{D}iv(fX)\, dM = \int_M \Big[g(\nabla f, X) + f\mathrm{D}iv(X)\Big]\, dM = 0 \tag{2.6}$$

With these notations fixed we have [19]

Theorem 2.2. *Let* (M,g) *be a Riemannian manifold. Then the following estimate holds*

$$\mu^*_p(M) \geq \sup\left\{\inf_\Omega\Big((1-p)\|X\|^q + \mathrm{D}iv(X)\Big),\ X \in \mathcal{W}^{1,1}(M)\right\} \tag{2.7}$$

When M is a Hadamard manifold, Mckean in [31], obtained an interesting bound for the fundamental tone for the laplacian (in our case, $p = 2$):

Theorem 2.3 (Mckean)**.** *Let* M *be an* n*-dimensional, complete noncompact, simply connected riemannian manifold with sectional curvature* $K \leq -c^2 < 0$*, then*

$$\lambda^*(M) \geq \frac{(n-1)^2 c^2}{4}.$$

As simple consequence of Theorem 2.1, we obtain a generalization of Mckean theorem

Theorem 2.4 (Generalized Mckean [19])**.** *Let M be an n-dimensional, complete noncompact, simply connected riemannian manifold with sectional curvature $K \leq -c^2 < 0$, then*

$$\mu_p^*(M) \geq \frac{(n-1)^p c^p}{p^p}.$$

In particular, when $p = 2$ this is Mckean theorem.

Contrary to the Laplace operator, the p-laplacian has not been proved to be discrete, even for Euclidean domains $\Omega \subset \mathbb{R}^n$ (as remarked in [28]). There are few results related to the spectrum of such operator. For instance, Lindqvist 1995, in [23], describes the first and the second eigenvalues for the p-laplacian. We would like to obtain other invariants which might provide us with some additional information relating the geometry of the manifold and its spectral structure. An interesting spectral invariant on M associated to the Laplace-Beltrami operator is the essential spectrum of M and the greatest lower bound of the essential spectrum of M, $\lambda_1^{ess}(M) = \lim_{j\to\infty} \lambda_1(M - K_j)$, where $K_1 \subset K_2 \subset \ldots$ is any exhaustion of M through compact subsets - this limit being independent on the exhaustion (see [5]). Due to the difficulties in the understanding the spectrum of the p-Laplace operator, we shall define its *essential p-first eigenvalue*, as

$$\mu_p^{ess}(M) := \lim_{j\to\infty} \mu_{1,p}(M - K_j), \tag{2.8}$$

where $K_1 \subset K_2 \subset \ldots$ is any exhaustion of M through compact subsets (more generally, we can define in a similar way its essential p-kth eigenvalue). With respect to essential spectrum we prove if $\theta(M)$ is the exponential volume growth of M defined by

$$\theta(M) = \limsup_{r\to\infty} \frac{1}{r} \log(V_r(x_0)) \tag{2.9}$$

where $V_r(x_0)$ is the volume of the geodesic ball $B_r(x_0)$, then we get a Brooks-type theorem (see [5])

Theorem 2.5. *If the volume of M is infinity, then $\mu_{1,p}^{ess}(M) \leq \frac{\theta(M)^p}{p^p}$.*

By the end of the year 2010 Professor Pessoa Lima told me to go to his office and called my attention to a paper of do Carmo and Zhou 1999 ([10]) where the technique of ODE of oscillation were used to obtain estimates for the first eingenvalue. As it is usual, he presented the interesting main results and showed how natural those results could be generalized to the setting of p-laplacian problems.

Those discussions turned into the paper ([20]) were we extend results obtained by Manfredo P. do Carmo and Detang Zhou ([10]), on estimation of the first eigenvalues for the Laplacian on complement of domains, precisely, we extend theorems 2.1 and 3.1 of [10]. The main tool used there is an oscilation theorem and the relationship between a Liouville equation and the associated Ricatti equation. Such a theorem can be naturally extended to the setup of the p-Laplacian:

Let $p > 1$ and denote $\Phi(s) = |s|^{p-2}s$. Below, p and q will represent conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$

Consider the Liouville type equation

$$(v(t)\Phi(x'(t)))' + \lambda v(t)\Phi(x(t)) = 0. \tag{2.10}$$

This equation is intimately related to the following Ricatti equation:

$$y' = \lambda v(t) + (p-1)v(t)^{1-q}y^q \tag{2.11}$$

Indeed, if $x(t)$ is a positive solution for (2.10), one set

$$y(t) := -\frac{v(t)\Phi(x'(t))}{\Phi(x(t))} \tag{2.12}$$

then, the first derivative of (2.12) gives:

$$\begin{aligned}
y'(t) &= -\frac{(v(t)\Phi(x'(t)))'}{\Phi(x(t))} + \frac{v(t)\Phi(x'(t))\Phi'(x(t))}{\Phi^2(x(t))} \\
&= \lambda v(t) + \frac{v(t)\Phi(x'(t))(p-1)x(t)^{p-2}x'(t)}{\Phi^2(x(t))} \\
&= \lambda v(t) + (p-1)v(t)\frac{|x'(t)|^p}{|x(t)|^p} \\
&= \lambda v(t) + (p-1)v(t)\frac{(|x'(t)|^{p-1})^{\frac{p}{p-1}}}{(|x(t)|^{p-1})^{\frac{p}{p-1}}} \\
&= \lambda v(t) + (p-1)\frac{y^q}{v(t)^{q-1}}
\end{aligned}$$

where, in the second equality we used equation (2.10).

We say that the equation (2.10) is oscillatory if a solution $x(t)$ for (2.10) has zeroes in $[T,\infty)$ for any $T \geq T_0$ (that is $x(t)$ has zeroes for arbitrarily large t). It is a classical result (see for instance [13]) that if one solution has arbitrarily large zeroes then, any other solution will also possess arbitrarily large zeroes, that is to be oscillatory is a characteristic of the equation and not to the particular solution.

Our main oscillation result is [20]:

Theorem 2.6 ([20], compare with theorem 2.1 [10]). *Let $v(t)$ be a positive continuous function on $[T_0, +\infty)$, and define*

$$V(t) := \int_{T_0}^{t} v(s)\, ds, \qquad \textit{and} \qquad \theta_V := \lim_{t\to\infty} \frac{\log V(t)}{t}$$

If the limit

$$\lim_{t\to\infty} V(t) = \int_{T_0}^{\infty} v(s)\, ds = +\infty$$

Then, equation (2.10) is oscillatory provided that

(a) $\lambda > 0 \quad \textit{and} \quad \theta_V = 0$

or else

(b) $\lambda > \dfrac{c^p}{p^p} \quad \textit{and} \quad \theta_V \le c$

As a particular case of item **(a)**, if $\lambda > 0$ and $V(t) \le At^c$, for certain positive constants, A, c, then

$$\theta_V = \lim_{t\to\infty} \frac{\log V(t)}{t} \le \lim_{t\to\infty} \frac{\log At^c}{t} = \lim_{t\to\infty} \left(\frac{\log A}{t} + \frac{c \log t}{t} \right) = 0$$

And, as a particular case of item **(b)**, if $\lambda > \dfrac{c^p}{p^p}$ and $V(t) \le Ae^{ct}$, then

$$\theta_V = \lim_{t\to\infty} \frac{\log V(t)}{t} \le \lim_{t\to\infty} \frac{\log Ae^{ct}}{t} = \lim_{t\to\infty} \left(\frac{\log A}{t} + \frac{ct}{t} \right) = c.$$

Let (M^n, g) be an open (that is complete, noncompact) Riemannian manifold and fix some base point $p \in M$. Denote by $v(r) := Area(\partial B_r(p))$ the area of the geodesic sphere of radius r centered at p. Put

$$V(r) := Vol(B_r(p)) = \int_0^r v(r)\, dr \quad \text{and set} \quad \theta(M) := \lim_{t\to\infty} \frac{\log V(t)}{t}.$$

Notice that the number $\theta(M)$ does not depend on the base point, p, so it is an invariant of the manifold that captures the growth behavior M off compact sets, since for any fixed $R > 0$ holds

$$\theta(M) = \lim_{t\to\infty} \frac{\log(V(R) + A(R,t))}{t} = \lim_{t\to\infty} \frac{\log A(R,t)}{t}$$

where $A(R,t) = V(t) - V(R)$. If M is a manifold with $Vol(M) = +\infty$ then, for any $T_0 > 0$, one has $\int_{T_0}^{+\infty} v(r)\, dr = +\infty$. Our main theorem is the following:

Theorem 2.7 ([20], compare with theorem 3.1 of [10]). *Let (M, g) be an open manifold with infinite volume, $Vol(M^n) = +\infty$. If $\Omega \subset M$ is an arbitrary compact set, denote by $\lambda_{1,p}(M - \Omega)$ be the first eigenvalue for the p-Laplacian on $M - \Omega$*

(a) *If M has subexponential growth, that is $\theta(M) = 0$, then $\lambda_{1,p}(M - \Omega) = 0$.*

(b) *If $\theta(M) \le \beta$, for some $\beta > 0$, then $\lambda_{1,p}(M - \Omega) \le \dfrac{\beta^p}{p^p}$.*

Remark 6. *As particular cases for this theorem we have:*

1. *If $Vol(B_r(p)) \le cr^\beta$ for some positive constants $c, \beta > 0$ and $r \ge r_0$, then $\lambda_{1,p}(M - \Omega) = 0$.*

2. *Let (M^n, o, g, k) denote a pointed Riemannian manifold, with a base point, $o \in M$, and $k : [0, \infty) \to \mathbf{R}$ a positive, nonincreasing function, with*

$$b_0(k) := \int_0^\infty sk(s)\, ds < \infty$$

such that the radial Ricci curvatures (that is, the Ricci curvatures along radial direction, ∂_r, tangent to minimal unitary geodesics departing from the base point) satisfy the bound

$$\mathrm{Ric}_M(\partial_r|_p) \geq -(n-1)k(dist(o,p)), \qquad \textit{for all } p \in M$$

we say that such a manifold has asymptotically nonnegative radial Ricci curvature, thus since for such a manifold one has $V(r) \leq e^{b_0(k)} r^n$ (see [34]), it follows, by the previous case that $\lambda_{1,p}(M-\Omega) = 0$.

3. *If $Vol(B_r(p)) \leq ce^{\beta t}$, then $\lambda_{1,p}(M-\Omega) \leq \dfrac{\beta^p}{p^p}$.*

We end up this section, setting the following question: what can be said about the spectrum for the following p-laplace problems on domains Ω of a manifold M^n (possibly with boundary $\partial M \neq \emptyset$):

No-flux problem

$$NF(\Omega): \quad \begin{cases} -\triangle_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = \text{constant} & \text{on } \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p-2} \dfrac{\partial u}{\partial \eta} = 0. & \end{cases}$$

Neumann problem

$$N(\Omega): \quad \begin{cases} -\triangle_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Robin problem

$$R(\Omega): \quad \begin{cases} -\triangle_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \eta} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Steklov problem

$$D(\Omega): \quad \begin{cases} \triangle_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta} + \lambda |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

2.2 Translational hypersurfaces

I was introduced to this quite interesting topic thanks to the very kind invitation from the friends Barnabé Lima, Juscelino Silva and Paulo A. Sousa who were studing some nice generalizations of the problem of obtaining examples of classes of minimal hypersurfaces. Although my contribution was rather insignificant, it brough me lots of new ideas related to this theme, opening a number of possibilities of new works related to translational surfaces. I am sincerely indebted for being invited to join the group. The discussion below is the introduction of our paper *Translation Hypersurfaces with Constant S_r Curvature in the Euclidean Space* [22]

It is well known that translation hypersurfaces are very important in Differential Geometry, providing an interesting class of constant mean curvature hypersurfaces and minimal hypersurfaces in a number of spaces endowed with good symmetries and even in certain applications in Microeconomics. There are many results about them, for instance, Chen et al. 2003 [8], Dillen et al. 1991 [9], Inoguchi et al. 2012 [16], Lima et al. 2014 [21], Liu 1999 [24], López 2011 [25], López and Moruz 2015 [26], López and Munteanu 2012 [27], Seo 2013 [33] and Chen 2011 [7], for an interesting application in Microeconomics.

Scherk 1835 [32] obtained the following classical theorem:

Let $M := \{(x,y,z) : z = f(x) + g(y)\}$ be a translation surface in $\mathbb{R}^3$, if is minimal then it must be a plane or the Scherk surface defined by

$$z(x,y) = \frac{1}{a} \ln \left| \frac{\cos(ay)}{\cos(ax)} \right|,$$

where a is a nonzero constant. In a different aspect, Liu 1999 [24] considered the translation surfaces with constant mean curvature in 3-dimensional Euclidean space and Lorentz-Minkowski space and Inoguchi et al. 2012 [16] characterized the minimal translation surfaces in the Heisenberg group Nil_3, and López and Munteanu [27], the minimal translation surfaces in Sol_3.

The concept of translation surfaces was also generalized to hypersurfaces of $\mathbb{R}^{n+1}$ by Dillen et al. 1991 [9], who obtained a classification of minimal translation hypersurfaces of the $(n+1)$-dimensional Euclidean space. A classification of the translation hypersurfaces with constant mean curvature in $(n+1)$-dimensional Euclidean space was made by Chen et al. 2003 [8].

The absence of an affine structure in hyperbolic space does not permit to give an intrinsic concept of translation surface as in the Euclidean setting. Considering the half-space model of hyperbolic space, López 2011 [25], introduced the concept of translation surface and presented a classification of the minimal translation surfaces. Seo 2013 [33] has generalized the results obtained by Lopez to the case of translation hypersurfaces of the $(n+1)$-dimensional hyperbolic space.

Definition 2.3. *We say that a hypersurface M^n of the Euclidean space $\mathbb{R}^{n+1}$ is a translation hypersurface if it is the graph of a function given by*

$$F(x_1,\ldots,x_n) = f_1(x_1) + \ldots + f_n(x_n)$$

where $(x_1,\ldots,x_n)$ *are cartesian coordinates and* f_i *is a smooth function of one real variable for* $i=1,\ldots,n$.

Now, let $M^n \subset \mathbb{R}^{n+1}$ be an oriented hypersurface and $\lambda_1,\ldots,\lambda_n$ denote the principal curvatures of M^n. For each $r=1,\ldots,n$, we can consider similar problems to the above ones, related with the r-th elementary symmetric polynomials, S_r, given by

$$S_r = \sum_{1\le i_1<\cdots<i_r\le n} \lambda_{i_1}\cdots\lambda_{i_r}$$

In particular, S_1 is the mean curvature, S_2 the scalar curvature and S_n the Gauss-Kronecker curvature, up to normalization factors. A very useful relationship involving the various S_r is given in the Proposition 1, Caminha 2006 [6].

Recently, some authors have studied the geometry of translational hypersurfaces under a condition in the S_r curvature, where $r>1$. Namely, Leite 1991 [18] gave a new example of a translation hypersurface of $\mathbb{R}^4$ with zero scalar curvature. Lima et al. 2014 presented a complete description of all translation hypersurfaces with zero scalar curvature in the Euclidean space $\mathbb{R}^{n+1}$ and Seo 2013 [33] proved that if M is a translation hypersurface with constant Gauss-Kronecker curvature GK in $\mathbb{R}^{n+1}$, then M is congruent to a cylinder, and hence $GK=0$.

The main results we obtained in the paper [22] were a complete classification of translation hypersurfaces of $\mathbb{R}^{n+1}$ with $S_r=0$ (that is, the r-minimal translation hypersurfaces). It is remarkable that one solution is a generalized Sherk hypersurface. Precisely:

Theorem 2.8 (Lima et al. 2016 [22]). *Let* M^n $(n\ge 3)$ *be a translation hypersurface in* $\mathbb{R}^{n+1}$. *Then, for* $2<r<n$, M^n *has zero* S_r

curvature if, and only if, it is congruent to the graph of the following functions

- $F(x_1,\dots,x_n) = \sum_{i=1}^{n-r+1} a_i x_i + \sum_{j=n-r+2}^{n} f_j(x_j) + b,$

on $\mathbb{R}^{n-r+1} \times J_{n-r+2} \times \cdots \times J_n$*, for some intervals* $J_{n-r+2},\dots,J_n$*, and arbitrary smooth functions* $f_i : J_i \subset \mathbb{R} \to \mathbb{R}$*. Which defines, after a suitable linear change of variables, a vertical cylinder, and*

- *A generalized periodic Enneper hypersurface given by*

$$F(x_1,\dots,x_n) = \sum_{k=n-r}^{n-1} \frac{\sqrt{\beta}}{a_k} \ln \left| \frac{\cos\left(-\dfrac{a_{n-r}\dots a_{n-1}}{\sigma_{r-1}(a_{n-r},\dots,a_{n-1})}\sqrt{\beta}x_n + b_n\right)}{\cos(a_k\sqrt{\beta}x_k + b_k)} \right| + \sum_{i=1}^{n-r-1} a_i x_i + c$$

on $\mathbb{R}^{n-r-1} \times I_{n-r} \times \cdots \times I_n$*, where* $a_1,\dots,a_{n-r},\dots,a_{n-1}$*,* $b_{n-r},\dots,b_n$ *and* c *are real constants where* $a_{n-r},\dots,a_{n-1}$ *and* $\sigma_{r-1}(a_{n-r},\dots,a_{n-1})$ *are nonzero,* $\beta = 1 + \sum_{i=1}^{n-r-1} a_i^2$*,* $I_k\,(n-r \le k \le n-1)$ *are open intervals defined by the conditions* $|a_k\sqrt{\beta}x_k + b_k| < \pi/2$ *while* I_n *is defined by*

$$\left| -\frac{a_{n-r}\dots a_{n-1}}{\sigma_{r-1}(a_{n-r},\dots,a_{n-1})}\sqrt{\beta}x_n + b_n \right| < \pi/2.$$

Theorem 2.9 (Lima et al. 2016 [22])**.** *Any translation hypersurface in* $\mathbb{R}^{n+1}$ $(n \ge 3)$ *with* S_r *constant, for* $2 < r < n$*, must have* $S_r = 0$*.*

Finally, we observe that, when one considers the upper half-space model of the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, that is,

$$\mathbb{R}^{n+1}_+ = \{(x_1,\dots,x_n,x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

endowed with the hyperbolic metric $ds^2 = \frac{1}{x_{n+1}^2}\left(dx_1^2 + \dots + dx_{n+1}^2\right)$ then, unlike in the Euclidean setting, the coordinates $x_1,\dots,x_n$ are interchangeable, but the same does not happen with the coordinate x_{n+1} and, due to this observation, López 2011 [25] and Seo 2013 [33] considered two classes of translation hypersurfaces in $\mathbb{H}^{n+1}$: *A hypersurface* $M \subset \mathbb{H}^{n+1}$ *is called a translation hypersurface of* **type I** (*respectively,* **type II**) *if it is given by an immersion* $X : U \subset \mathbb{R}^n \to \mathbb{H}^{n+1}$ *satisfying*

$$X(x_1,\dots,x_n) = (x_1,\dots,x_n, f_1(x_1) + \dots + f_n(x_n))$$

where each f_i *is a smooth function of a single variable. Respectively, in case of* **type II**,

$$X(x_1,\dots,x_n) = (x_1,\dots,x_{n-1}, f_1(x_1) + \dots + f_n(x_n), x_n)$$

Seo proved

Theorem 2.10 (Theorem 3.2, Seo 2013 [33])**.** *There is no minimal translation hypersurface of* **type I** *in* $\mathbb{H}^{n+1}$.

and with respect to **type II** surfaces he proved

Theorem 2.11 (Theorem 3.3, Seo 2013)**.** *Let* $M \subset \mathbb{H}^3$ *be a minimal translation surface of* **type II** *given by the parametrization* $X(x,z) = (x, f(x) + g(z), z)$. *Then the functions* f *and* g *are as follows:*

$$\begin{aligned} f(x) &= ax + b, \\ g(z) &= \sqrt{1+a^2}\int \frac{cz^2}{\sqrt{1-c^2z^4}}\,dz, \end{aligned}$$

where a, b, *and* c *are constants.*

We emphasize that the result proved by Seo, Theorem 3.2 of Seo 2013, implies that our result (Theorem 2.9) is not valid in the hyperbolic space context.

We finish this short presentation setting some questions related to translation hypersurfaces:

Question I Obtain a complete classification of r-minimal translation hypersurfaces in Lorentzian spaces;

Question II Define translation surfaces in Lie groups and obtain a classification of minimal translation surfaces in certain classes of these spaces.

Question III Is it possible to define translation surfaces in homogeneous spaces? If the answer is positive, obtain examples or classification.

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Spectral estimates for the L_Φ-operator and the p-Laplacian

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In honor of Professor Barnabé Pessoal Lima on the occasion of his 60th birthday.

Abstract: The purpose of this article is to present part of the work of Professor Barnabé Pessoa Lima on eigenvalue estimates on manifolds. It is based on the articles [7] and [27]. The work presented here shows clever and creative insights that is his "trademark" when discussing mathematics.

1 Introduction

Let M be a connected Riemannian manifold, possibly incomplete, and let $\triangle = \operatorname{div} \circ \nabla$ be the Laplace-Beltrami operator on acting on $C_0^\infty(M)$, the space of smooth functions with compact support. When M is geodesically complete, $\triangle$ is essentially self-adjoint, thus there is a unique self-adjoint extension to an unbounded operator, denoted by $\triangle$, whose domain is the set of functions $f \in L^2(M)$ so that $\triangle f \in L^2(M)$, see [15], [16] and [34]. If M is not complete we will always consider the Friedrichs extension of $\triangle$. The spectrum of $\triangle$ denoted by $\sigma(\triangle) \subset [0, \infty)$, is formed by

†Partially suported by CNPq and FUNCAP/CNPq/PRONEX

all $\lambda \in \mathbb{R}$ for which $\triangle + \lambda I$ is not injective or the inverse operator $(\triangle + \lambda I)^{-1}$ is unbounded, [16]. Is well known that the bottom of the spectrum $\inf \sigma(M)$ is equal to the fundamental tone $\lambda^*(M)$ and given by

$$\lambda^*(M) = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2}; f \in C_0^\infty(M) \backslash \{0\} \right\}.$$

The relations between the $\triangle$-fundamental tone Riemannian manifolds and their geometric invariants has been the subject to an intensive research in the past 50 years. There is a huge literature on this subject, however we limit ourselves to quote the classics [5], [6], [12] and references therein for a detailed picture. One can consider elliptic operators more general than the Laplace-Beltrami operator. Indeed, let $\Omega \subset M$ be a connected subset of a Riemannian manifold and $\Phi : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of $T\Omega$. There exists a second order elliptic operator associated to the section Φ, acting of smooth functions, namely the operator $L_\Phi(f) = \mathrm{div}(\Phi \nabla f)$, $f \in C^\infty(\Omega)$. This class of operators include the Laplace-Beltrami operator $\triangle$, more precisely, when Φ is the identity section then $L_\Phi = \triangle$, is the Laplace operator. The L_Φ-fundamental tone of Ω is defined by

$$\lambda^{L_\Phi}(\Omega) = \inf \left\{ \frac{\int_\Omega |\Phi^{1/2} \nabla f|^2}{\int_\Omega f^2}; f \in C_0^2(\Omega) \setminus \{0\} \right\}. \tag{1.1}$$

The method developed in [9] for giving lower bounds for the $\triangle$-fundamental tone established was extended for self-adjoint elliptic operators L_Φ by Bessa et al in [7]. There, they proved the following theorem.

Theorem 1.1 (Bessa-Jorge-Lima-Montenegro). *Let $\Omega \subset M$ be a domain in a Riemannian manifold M and let $\Phi : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of $T\Omega$. Then the L_Φ-fundamental tone of Ω has the following lower bound*

$$\lambda^{L_\Phi}(\Omega) \geq \sup_{\mathcal{X}(\Omega)} \inf_{\Omega} \left[\mathrm{div}(\Phi X) - |\Phi^{1/2} X|^2 \right]. \tag{1.2}$$

If Ω is bounded and with smooth boundary $\partial\Omega \neq \varnothing$ then we have equality in (1.2).

$$\lambda^{L_\Phi}(\Omega) \quad = \quad \sup_{\mathcal{X}(\Omega)} \inf_{\Omega} \left[\mathrm{div}(\Phi X) - |\Phi^{1/2} X|^2 \right]. \tag{1.3}$$

Where $\mathcal{X}(\Omega)$ is the set of all smooth vector fields on Ω.

Recently, another operator, called p-Laplacian, arising from problems on non-newtonian fluids, glaceology, nonlinear elasticity, and in problems of nonlinear partial differential equations came to light. For $1 < p < \infty$, the p-Laplacian on Ω is defined by

$$\triangle_p(u) = -\mathrm{div}(|\nabla u|^{p-2}(\nabla u)). \tag{1.4}$$

The p-Laplacian appears naturally in the variational problem associated to the energy functional $E_p : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$E_p(u) = \int_\Omega |\nabla u|^p \, d\nu, \tag{1.5}$$

where $W_0^{1,p}(\Omega)$ denotes the Sobolev space given by the closure of $C_0^\infty(\Omega)$ functions with compact support in Ω for the norm

$$\|u\|_{1,p}^p = \int_\Omega |u|^p \, d\nu + \int_\Omega |\nabla u|^p \, d\nu.$$

Observe that, when $p=2$, $\triangle_2$ is the Laplace-Beltrami operator. The interesting problem here is the non-linear "eigenvalue" problem

$$\triangle_p u + \lambda |u|^{p-2}u = 0$$

The solutions for this problem, for arbitrary $p \in (1,\infty)$ must be described in the sense of distributions, that is, $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ is an eigenfunction associated to the eigenvalue λ, if

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle\, dv = \lambda \int_\Omega |u|^{p-2}u\phi\, dv \tag{1.6}$$

for any test function $\phi \in C_0^\infty(\Omega)$. The smallest λ for which the there is $u \in W_0^{1,p}(\Omega)$ satisfying (1.6) is the p-fundamental tone $\lambda_p^*(\Omega)$ of Ω defined by

$$\lambda_p^*(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla u|^p dv}{\int_\Omega u^p dv}, u \in W_0^{1,p}(\Omega)\setminus\{0\}\right\}. \tag{1.7}$$

Definition 1.1. *Let $\Omega \subset M$ be a open subset of a Riemannian manifold M and vector field $X \in L^1_{loc}(\chi(M))$, in the sense that $|X| \in L^1(\Omega)$. A function $h \in L^1_{loc}(\Omega)$ is said to be a weak divergence of X, denoted by $h = \mathrm{div} X$ if for every $\phi \in C_0^\infty(\Omega)$ then*

$$\int_\Omega h\phi\, dv = -\int_\Omega \langle \nabla\phi, X\rangle\, dv \tag{1.8}$$

The weak divergence exists for almost every point of Ω. If $X \in \mathcal{W}^{1,1}(\Omega)$ and $f \in C_0^1(\Omega)$ then $fX \in \mathcal{W}^{1,1}(\Omega)$ with $\mathrm{Div}(fX) = g(\nabla f, X) + f\mathrm{Div}X$. In particular for $f \in C_0^\infty(\Omega)$ we have that

$$\int_M \mathrm{Div}(fX)\, d\mu = \int_M \left[g(\nabla f, X) + f\mathrm{Div}(X)\right] d\mu = 0 \tag{1.9}$$

Definition 1.2. *Let Ω be an open subset of a Riemannian man-*

ifold M. Let $\mathfrak{X}(\Omega)$ be the set of all smooth bounded vector fields on Ω with $\inf_\Omega \operatorname{div} X > 0$. *Define $c(\Omega)$ by*

$$c(\Omega) := \sup\left\{\frac{\inf_\Omega \operatorname{div} X}{\|X\|_\infty};\ X \in \mathfrak{X}(\Omega)\right\}$$

B. Lima et al. in [27], proved the following two estimates, Theorems 1.2 and 1.3 for the smallest "eigenvalue"$\lambda_p^*(\Omega)$ of the p-Laplacian.

Theorem 1.2 (Lima-Montenegro-Santos)**.** *Let $\Omega \subset M$ be an open subset of a Riemannian manifold, M. Then*

$$\lambda_p^*(\Omega) \geq \frac{c(\Omega)^p}{p^p}. \tag{1.10}$$

Corollary 1.1 (Generalized Mckean)**.** *Let M be an n-dimensional, complete noncompact, simply connected Riemannian manifold with sectional curvature $K \leq -1 < 0$, then*

$$\lambda_p^*(M) \geq \frac{(n-1)^p}{p^p}.$$

When $p = 2$ this is the McKean theorem [28]. A second estimate proven in [27] was the following version for the p-Laplacian of the main result of [9].

Theorem 1.3 (Lima-Montenegro-Santos)**.** *Let $\Omega \subset M$ be an open subset of a Riemannian manifold M. Then the following estimate holds*

$$\lambda_p^*(M) \geq \sup_{X \in \chi(\Omega)} \left\{\inf_\Omega \left((1-p)|X|^q + \operatorname{Div}(X)\right),\ X \in W^{1,1}(M)\right\} \tag{1.11}$$

This article surveys the work of Barnabé Pessoa Lima on eigenvalue estimates. It is based on [7] and [27]. The main results, Theorems (1.1), (1.2) and (1.3) has various geometric applications we describe in the next sections.

2 Geometric applications of Theorem 1.1

Consider the linearized operator L_r of the $(r+1)$-mean curvature

$$H_{r+1} = S_{r+1}/\binom{n}{r+1}$$

arising from normal variations of a hypersurface M immersed into the $(n+1)$-dimensional simply connected space form $\mathbb{N}^{n+1}(c)$ of constant sectional curvature $c \in \{1,0,-1\}$ where S_{r+1} is the $(r+1)$-th elementary symmetric function of the principal curvatures $k_1, k_2, \ldots, k_n$. Recall that the elementary symmetric function of the principal curvatures are given by

$$S_0 = 1, \quad S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad 1 \le r \le n. \tag{2.1}$$

Letting $A = -(\overline{\nabla}\eta)$ be the shape operator of M, where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{N}^{n+1}(c)$ and η a globally defined unit vector field normal to M, we can recursively define smooth symmetric sections $P_r : M \to \mathrm{End}(TM)$, for $r = 0, 1, \ldots, n$, called the Newton operators, setting $P_0 = I$ and $P_r = S_r Id - AP_{r-1}$ so that $P_r(x) : T_xM \to T_xM$ is a self-adjoint linear operator with the same eigenvectors as the shape operator A. The operator L_r is the second order self-adjoint differential operator

$$L_{P_r}(f) = \mathrm{div}(P_r \nabla f) \tag{2.2}$$

associated to the section P_r. However, the sections P_r may be not positive definite and then the operators L_r may not be elliptic, see [29]. However, there are geometric hypothesis that imply the ellipticity of L_r, see [11], [24], [4]. Here we will not impose geometric conditions to guarantee ellipticity of the L_r, except in corollary (2.2). Instead we will ask the ellipticity on the set of hypothesis in the following way. It is known, see [23], that there is an open and dense subset $U \subset M$ where the ordered eigenvalues $\{\mu_1^r(x) \leq \ldots \leq \mu_n^r(x)\}$ of $P_r(x)$ depend smoothly on $x \in U$ and continuously on $x \in M$. In addition, the respective eigenvectors $\{e_1(x), \ldots, e_n(x)\}$ form a smooth orthonormal frame in a neighborhood of every point of U. Set $\nu(P_r) = \sup_{x \in M}\{\mu_n^r(x)\}$ and $\mu(P_r) = \inf_{x \in M}\{\mu_1^r(x)\}$. Observe that if $\mu(P_r) > 0$ then P_r is positive definite, thus L_r is elliptic. The following definition of locally bounded $(r+1)$-th mean curvature hypersurface is necessary in order to state the next result.

Definition 2.1. *An oriented immersed hypersurface $\varphi : M \hookrightarrow N$ of a Riemannian manifold N is said to have locally bounded $(r+1)$-th mean curvature H_{r+1} if for any $p \in N$ and $R > 0$, the number $h_{r+1}(p,R) = \sup\{|S_{r+1}(x)| = a(n,r+1) \cdot |H_{r+1}(x)|; x \in \varphi(M) \cap B_N(p,R)\}$ is finite. Here $B_N(p,R) \subset N$ is the geodesic ball of radius R and center $p \in N$.*

The next result generalizes in some aspects the main application of [8]. There the first and fourth authors give lower bounds for $\triangle$-fundamental tone of domains in submanifolds with locally bounded mean curvature in complete Riemannian manifolds.

Theorem 2.1. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ be an oriented hypersurface immersed with locally bounded $(r+1)$-th mean curvature H_{r+1} for* *some $r \leq n-1$ and with $\mu(P_r) > 0$. Let $B_{\mathbb{N}^{n+1}(c)}(p,R)$ be the geodesic ball centered at $p \in \mathbb{N}^{n+1}(c)$ with radius R and*

$$\Omega \subset \varphi^{-1}(\overline{B_{\mathbb{N}^{n+1}(c)}(p,R)})$$

be a connected component. Then the L_r-fundamental tone $\lambda^{L_r}(\Omega)$ of Ω has the following lower bounds.

i. *For $c = 1$ and $0 < R < \cot^{-1}\left[\dfrac{(r+1)\cdot h_{r+1}(p,R)}{(n-r)\cdot \inf_\Omega S_r}\right]$ we have that*

$$\lambda^{L_r}(\Omega) \geq 2\cdot\frac{1}{R}\Big[(n-r)\cdot\cot[R]\cdot\inf_\Omega S_r - (r+1)\cdot h_{r+1}(p,R)\Big]. \quad (2.3)$$

ii. *For $c \leq 0$, $h_{r+1}(p,R) \neq 0$ and $0 < R < \dfrac{(n-r)\cdot\inf_\Omega S_r}{(r+1)\cdot h_{r+1}(p,R)}$ we have that*

$$\lambda^{L_r}(\Omega) \geq 2\cdot\frac{1}{R^2}\Big[(n-r)\cdot\inf_\Omega S_r - (r+1)\cdot R\cdot h_{r+1}(p,R)\Big]. \quad (2.4)$$

iii. *If $c \leq 0$, $h_{r+1}(p,R) = 0$ and $R > 0$ we have that*

$$\lambda^{L_r}(\Omega) \geq \frac{2(n-r)\inf_\Omega S_r}{R^2} \quad (2.5)$$

Definition 2.2. *Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a closed Riemannian manifold into a complete Riemannian manifold N. For each $x \in N$, let $r(x) = \sup_{y\in M} dist_N(x,\varphi(y))$. The extrinsic radius $R_e(M)$ of M is defined by*

$$R_e(M) = \inf_{x\in N} r(x).$$

Moreover, there is a point $x_0 \in N$ called the barycenter of $\varphi(M)$ in N such that $R_e(M) = r(x_0)$.

Corollary 2.1. *Let $\varphi : M \hookrightarrow B_{\mathbb{N}^{n+1}(c)}(R) \subset \mathbb{N}^{n+1}(c)$ be a complete oriented hypersurface with bounded $(r+1)$-th mean curvature H_{r+1} for some $r \leq n-1$, R chosen as in Theorem (2.1). Suppose that $\mu(P_r) > 0$ so that the L_r operator is elliptic. Then M is not closed.*

Corollary 2.2. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)^2$, $c \in \{1, 0, -1\}$ be an oriented closed hypersurface with $H_{r+1} > 0$. Then there is an explicit constant $\Lambda_r = \Lambda_r(c, \inf_M S_r, \sup_M S_{r+1}) > 0$ such that the extrinsic radius $R_e(M) \geq \Lambda_r$.*

i. For $c = 1$, $\Lambda_r = \cot^{-1}\left[\dfrac{(r+1)\cdot \sup_M S_{r+1}}{(n-r)\cdot \inf_\Omega S_r}\right]$.

ii. For $c \in \{0, -1\}$, $\Lambda_r = \dfrac{(n-r)\cdot \inf_\Omega S_r}{(r+1)\cdot \sup_\Omega S_{r+1}}$.

Remark 7. *The hypothesis H_{r+1} implies that $H_j > 0$ and L_j are elliptic for $j = 0, 1, \ldots r$, see [4], [11] or [24]. Thus in fact in fact have that $R_e \geq \max\{\Lambda_0, \cdots, \Lambda_r\}$.*

Remark 8. *Jorge and Xavier, (Theorem 1 of [22]), proved the inequalities of Corollary (2.2) when $r = 0$ for complete submanifolds with scalar curvature bounded from below contained in a compact ball of a complete Riemannian manifold. Moreover, for $c = -1$ their inequality is slightly better. These inequalities should be also compared with a related result proved by Fontenele-Silva in [17].*

Corollary 2.3. *Let $\varphi : M \hookrightarrow \mathbb{S}^{n+1}(1)$, be an oriented closed hypersurface with $\mu_1^r(M) > 0$ and $H_{r+1} = 0$. Then the extrinsic radius $R_e(M) \geq \pi/2$.*

Remark 9. *An interesting question is: Is it true that any closed oriented hypersurface with $\mu_1^r(M) > 0$ and $H_{r+1} = 0$ intersect every great circle? For $r = 0$ it is true and it was proved by Frankel [18].*

[2] If $c = 1$ suppose that $\mathbb{N}^{n+1}(c)$ is the open hemisphere of $\mathbb{S}^{n+1}_+$.

We now consider immersed hypersurfaces $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ with L_r and L_s elliptic. We can compare the L_r and L_s fundamental tones of a domain $\Omega \subset M$. In particular we can compare with its L_0-fundamental tone.

Theorem 2.2. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ be an oriented n-dimensional hypersurface M immersed into the $(n+1)$-dimensional simply connected space form of constant sectional curvature c and $\mu(L_r) > 0$ and $\mu(L_s) > 0$, $0 \leq s, r \leq n-1$. Let $\Omega \subset M$ be a domain with compact closure and piecewise smooth non-empty boundary. Then the L_r and L_s fundamental tones satisfies the following inequalities*

$$\lambda^{L_r}(\Omega) \geq \frac{\mu(P_r)}{\nu(P_s)} \cdot \lambda^{L_s}(\Omega), \tag{2.6}$$

where $\lambda^{L_s}(\Omega)$ and $\lambda^{L_r}(\Omega)$ are respectively the first L_s-eigenvalue and L_r-eigenvalue of Ω. From (2.6) we have in particular that

$$\nu(r) \cdot \lambda^{\triangle}(\Omega) \geq \lambda^{L_r}(\Omega) \geq \mu(r) \cdot \lambda^{\triangle}(\Omega) \tag{2.7}$$

2.1 Closed eigenvalue problem

Let M be a closed hypersurface of a simply connected space form $\mathbb{N}^{n+1}(c)$. Similarly to the eigenvalue problem of closed Riemannian manifolds, the interesting problem is what bounds can one obtain for the first nonzero L_r-eigenvalue $\lambda_1^{L_r}(M)$ in terms of the geometries of M and of the ambient space. Upper bounds for the first nonzero $\triangle$-eigenvalue or even for the first nonzero L_r-eigenvalue, $r \geq 1$ have been obtained by many authors in contrast with lower bounds that are rare. For instance, Reilly [30] extending earlier result of Bleecker and Weiner [10] obtained upper bounds for $\lambda_1^{\triangle}(M)$ of a closed submanifold M of $\mathbb{R}^m$ in terms of the total mean curvature of M. Reilly's result applied to compact submanifolds of the sphere $M \subset \mathbb{S}^{m+1}(1)$, this later viewed as a hypersurface of the Euclidean space $\mathbb{S}^{m+1}(1) \subset \mathbb{R}^{m+2}$ obtains upper bounds for $\lambda_1^{\triangle}(M)$, see [2]. Heintze,[20] extended Reilly's result to compact manifolds and Hadamard manifolds $\overline{M}$. In particular for the hyperbolic space $\mathbb{H}^{n+1}$. The best upper bounds for the first nonzero $\triangle$-eigenvalue of closed hypersurfaces M of $\mathbb{H}^{n+1}$ in terms of the total mean curvature of M was obtained by El Soufi and Ilias [32]. Regarding the L_r operators, Alencar, Do Carmo, and Rosenberg [2] obtained sharp (extrinsic) upper bound the first nonzero eigenvalue $\lambda_1^{L_r}(M)$ of the linearized operator L_r of compact hypersurfaces M of $\mathbb{R}^{m+1}$ with $S_{r+1} > 0$. Upper bounds for $\lambda_1^{L_r}(M)$ of compact hypersurfaces of $\mathbb{S}^{n+1}$, $\mathbb{H}^{n+1}$ under the hypothesis that L_r is elliptic were obtained by Alencar, Do Carmo, Marques in [1] and by Alias and Malacarne in [3] see also the work of Veeravalli [35]. On the other hand, lower bounds for $\lambda_1^{L_r}(M)$ of closed hypersurfaces $M \subset \mathbb{N}^{n+1}(c)$ are not so well studied as the upper bounds, except for $r = 0$ in which case $L_0 = \triangle$. In this paper we make a simple observation (Theorem 2.3) that to obtain lower and upper bounds for the L_Φ-eigenvalues (Dirichlet or Closed eigenvalue problem) it is enough to obtain lower and upper bounds for the eigenvalues of Φ and for the eigenvalues for the Laplacian in the respective problem. When applied to the L_r operators (supposing them elliptic) we obtain lower bounds for closed hypersurfaces of the space forms via Cheeger's lower bounds for the first $\triangle$-eigenvalue of closed manifolds. Let $\{\mu_1(x) \leq \ldots \leq \mu_n(x)\}$ be the ordered eigenvalues of $\Phi(x)$. Setting $\nu(\Phi) = \sup_{x \in \Omega}\{\mu_n(x)\}$ and $\mu(\Phi) = \inf_{x \in \Omega}\{\mu_1(x)\}$ we have the following theorem.

Theorem 2.3. *Let $\lambda^{L_\Phi}(\Omega)$ denote the L_Φ-fundamental tone of Ω if Ω is unbounded or $\partial\Omega \neq \emptyset$ and the first nonzero L_Φ-eigenvalue $\lambda_1^{L_\Phi}(\Omega)$ if Ω is a closed manifold. Then $\lambda^{L_\Phi}(\Omega)$ satisfies the following inequalities,*

$$\nu(\Phi,\Omega)\cdot\lambda^{\triangle}(\Omega) \geq \lambda^{L_\Phi}(\Omega) \geq \mu(\Phi,\Omega)\cdot\lambda^{\triangle}(\Omega), \tag{2.8}$$

where $\lambda^{\triangle}(\Omega)$ is the $\triangle$-fundamental tone of Ω or the first nonzero $\triangle$-eigenvalue of Ω.

Let M be a closed n-dimensional Riemannian manifold, in [13] Cheeger defined the following constant given by

$$h(M) = \inf_S \frac{vol_{n-1}(S)}{\min\{vol_n(\Omega_1), vol_n(\Omega_2)\}}, \tag{2.9}$$

where $S \subset M$ ranges over all connected closed hypersurfaces dividing M in two connected components, i.e. $M = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ such that $S = \partial\Omega_1 = \partial\Omega_2$ and he proved that the first nonzero $\triangle$-eigenvalue $\lambda_1^{\triangle}(M) \geq h(M)^2/4$.

Corollary 2.4. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$, $c \in \{1,0,-1\}$*[3] *be an oriented closed hypersurface with $H_{r+1} > 0$. Then the first nonzero L_r-eigenvalue of M has the following lower bound*

$$\lambda_1^{L_r}(M) \geq \mu(L_r)\cdot\frac{h^2(M)}{4}.$$

3 Proof of Theorem 1.1 and its geometric applications

Let Ω be an arbitrary domain, X be a smooth vector field on

[3] If $c = 1$ suppose that $\mathbb{N}^{n+1}(c)$ is the open hemisphere of $\mathbb{S}^{n+1}_+$.

Ω and $f \in C_0^\infty(\Omega)$. The vector field $f^2\Phi X$ has compact support $\mathrm{supp}(f^2\Phi X) \subset \mathrm{supp}(f) \subset \Omega$. Let $\mathcal{S}$ be a regular domain containing the support of f. We have by the divergence theorem that

$$\begin{aligned} 0 &= \int_{\mathcal{S}} \mathrm{div}(f^2\Phi X) = \int_\Omega \mathrm{div}(f^2\Phi X) \\ &= \int_\Omega \left[\langle \nabla f^2, \Phi X\rangle + f^2 \mathrm{div}(\Phi X)\right] \end{aligned}$$

and hence,

$$\begin{aligned} 0 &= \int_\Omega \left[\langle \nabla f^2, \Phi X\rangle + f^2 \mathrm{div}(\Phi X)\right] \\ &\geq -2\int_\Omega \left[|f|\cdot|\Phi^{1/2}\nabla f|\cdot|\Phi^{1/2}X| + \mathrm{div}(\Phi X)\cdot f^2\right] \\ &\geq \int_\Omega \left[-|\Phi^{1/2}\nabla f|^2 - f^2\cdot|\Phi^{1/2}X|^2 + \mathrm{div}(\Phi X)\cdot f^2\right]. \end{aligned}$$

Therefore

$$\begin{aligned} \int_\Omega |\Phi^{1/2}\nabla f|^2 &\geq \int_\Omega \left[\mathrm{div}(\Phi X) - |\Phi^{1/2}X|^2\right] f^2 \\ &\geq \inf\left[\mathrm{div}(\Phi X) - |\Phi^{1/2}X|^2\right]\int_\Omega f^2. \end{aligned} \tag{3.1}$$

By the variational formulation (1.1) of $\lambda^{L_r}(\Omega)$ this inequality above implies that

$$\lambda^{L_r}(\Omega) \geq \inf_\Omega\left[\mathrm{div}(\Phi X) - |\Phi^{1/2}X|^2\right]. \tag{3.2}$$

When Ω is a bounded domain with smooth boundary $\partial\Omega \neq \emptyset$ then $\lambda^{L_r}(\Omega) = \lambda_1^{L_r}(\Omega)$. This proof above shows that

$$\lambda_1^{L_r}(M) \geq \inf_M\left[\mathrm{div}(\Phi X) - |\Phi^{1/2}X|^2\right].$$

Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a positive first L_r-eigenfunction[4] of Ω and if we set $X_0 = -\nabla \log(v)$ we have that

$$
\begin{aligned}
\operatorname{div}(\Phi X_0) - |\Phi^{1/2} X_0|^2 &= -\operatorname{div}\left((1/v)\,\Phi\,\nabla v\right) - (1/v^2)\,|\Phi^{1/2}\,\nabla v|^2 \\
&= (1/v^2)\langle \nabla v, \Phi\,\nabla v\rangle - (1/v)\operatorname{div}(\Phi\,\nabla v) \\
&\quad -(1/v^2)\,|\Phi^{1/2}\,\nabla v|^2 \\
&= -(1/v)\operatorname{div}(\Phi\,\nabla v) = -L_r(v)/v = \lambda_1^L(\Omega).
\end{aligned}
$$

This proves (1.3).

3.1 Proof of Theorem 2.1 and Corollaries 2.1, 2.2, 2.3

We start this section stating few lemmas necessary to construct the proof of Theorem (2.1). The first lemma was proved in [21] for the Laplace operator and for the L_r operator in [25] and [26]. We reproduce its proof to make the exposition complete.

Lemma 3.1. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ be a complete hypersurface immersed in $(n+1)$-dimensional simply connected space form $\mathbb{N}^{n+1}(c)$ of constant sectional curvature c. Let $g : \mathbb{N}^{n+1}(c) \to \mathbb{R}$ be a smooth function and set $f = g \circ \varphi$. Identify $X \in T_pM$ with $d\varphi(p)X \in T_{\varphi(p)}\varphi(M)$ then we have that*

$$
L_r f(p) \;=\; \textstyle\sum_{i=1}^{n} \mu_i^r \operatorname{Hess} g(\varphi(p))\,(e_i, e_i) + \operatorname{Trace}(AP_r)\langle \nabla g, \eta\rangle. \tag{3.3}
$$

Proof: Each P_r is also associated to a second order self-adjoint differential operator defined by $\square f = \operatorname{Trace}(P_r \operatorname{Hess}(f))$ see [14], [19]. We have that

$$
\square f = \operatorname{Trace}(P_r \operatorname{Hess}(f)) = \operatorname{div}(P_r\,\nabla f) - \operatorname{trace}(\nabla P_r)\,\nabla f. \tag{3.4}
$$

[4] $v \in C^2(\Omega) \cap H_1^0(\Omega)$ if $\partial\Omega$ is not smooth.

Rosenberg [31] proved that when the ambient manifold is the simply connected space form $\mathbb{N}^{n+1}(c)$ then $\mathrm{Trace}(\nabla P_r)\,\nabla \equiv 0$, see also [29]. Therefore $L_r f = \mathrm{Trace}(P_r \mathrm{Hess}(f))$. Using Gauss equation to compute $\mathrm{Hess}(f)$ we obtain

$$\mathrm{Hess}\, f(p)(X,Y) = \mathrm{Hess}\, g(\varphi(p))(X,Y) + \langle \nabla g, \alpha(X,Y)\rangle_{\varphi(p)}, \tag{3.5}$$

where $\langle \alpha(X,Y),\eta\rangle = \langle A(X),Y\rangle$. Let $\{e_i\}$ be an orthonormal frame around p that diagonalize the section P_r so that $P_r(x)(e_i) = \mu_i^r(x)e_i$. Thus

$$\begin{aligned} L_r f &= \textstyle\sum_{i=1}^n \langle P_r \mathrm{Hess}\, f(e_i), e_i\rangle \\ &= \textstyle\sum_{i=1}^n \langle \mathrm{Hess}\, f(e_i), \mu_i^r e_i\rangle \\ &= \textstyle\sum_{i=1}^n \mu_i^r\, \mathrm{Hess}\, f(e_i,e_i) \end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6) we have that

$$\begin{aligned} L_r f &= \textstyle\sum_{i=1}^n \mu_i^r\, \mathrm{Hess}\, g(e_i,e_i) + \langle \nabla g, \sum_{i=1}^n \mu_i^r \alpha(e_i,e_i)\rangle \\ &= \textstyle\sum_{i=1}^n \mu_i^r\, \mathrm{Hess}\, g(e_i,e_i) + \langle \nabla g, \alpha(\sum_{i=1}^n P_r(e_i), e_i)\rangle \\ &= \textstyle\sum_{i=1}^n \mu_i^r\, \mathrm{Hess}\, g(e_i,e_i) + \mathrm{Trace}(AP_r)\langle \nabla g, \eta\rangle \end{aligned} \tag{3.7}$$

Here $\mathrm{Hess}\, f(X) = \nabla_X \nabla f$ and $\mathrm{Hess}\, f(X,Y) = \langle \nabla_X \nabla f, Y\rangle$. The next two lemmas we are gong to present are well known and their proofs are easily found in the literature thus we will omit them here.

Lemma 3.2 (Hessian Comparison Theorem)**.** *Let M be a complete Riemannian manifold and $x_0, x_1 \in M$. Let $\gamma : [0, \rho(x_1)] \to M$ be a minimizing geodesic joining x_0 and x_1 where $\rho(x)$ is the distance function $dist_M(x_0,x)$. Let K be the sectional curvatures of M and $\upsilon(\rho)$, defined below.*

$$\upsilon(\rho) = \begin{cases} k_1 \cdot \coth(k_1 \cdot \rho(x)), & if \quad \sup_\gamma K = -k_1^2 \\ \dfrac{1}{\rho(x)}, & if \quad \sup_\gamma K = 0 \\ k_1 \cdot \cot(k_1 \cdot \rho(x)), & if \quad \sup_\gamma K = k_1^2 \text{ and } \rho < \pi/2k_1. \end{cases} \tag{3.8}$$

Let $X = X^\perp + X^T \in T_xM$*,* $X^T = \langle X, \gamma' \rangle \gamma'$ *and* $\langle X^\perp, \gamma' \rangle = 0$*. Then*

$$Hess\,\rho(x)(X,X) = Hess\,\rho(x)(X^\perp,X^\perp) \geq \upsilon(\rho(x)) \cdot \|X^\perp\|^2. \tag{3.9}$$

See [33] for a proof.

Lemma 3.3. *Let* $p \in M$ *and* $1 \leq r \leq n-1$*, let* $\{e_i\}$ *be an orthonormal basis of* T_pM *such that* $P_r(e_i) = \mu_i^r e_i$ *and* $A(e_i) = k_i e_i$*. Then*

i. $\operatorname{trace}(P_r) = \sum_{i=1}^n \mu_i^r = (n-r)S_r$

ii. $\operatorname{trace}(AP_r) = \sum_{i=1}^n k_i \mu_i^r = (r+1)S_{r+1}$

In particular, if the Newton operator P_r *is positive definite then* $S_r > 0$*.*

To prove Theorem (2.1) set $g : B(p,R) \subset \mathbb{N}^{n+1}(c) \to \mathbb{R}$ given by $g = R^2 - \rho^2$, where ρ is the distance function ($\rho(x) = \operatorname{dist}(x,p)$) of $\mathbb{N}^{n+1}(c)$. Setting $f = g \circ \varphi$ we obtain by (3.3) that

$$L_r f = \sum_{i=1}^n \mu_i^r \cdot \operatorname{Hess} g(e_i, e_i) + (r+1) \cdot S_{r+1} \cdot \langle \nabla g, \eta \rangle, \tag{3.10}$$

since $\operatorname{Trace}(AP_r) = (r+1) \cdot S_{r+1}$. Letting $X = -\nabla \log f$ we have by Theorem (1.1) that

$$\begin{aligned} \lambda^{L_r}(\Omega) &\geq \inf_\Omega(-L_r f/f) \\ &= \inf_\Omega \left\{ -\frac{1}{g} \left[\sum_{i=1}^n \mu_i^r \cdot \operatorname{Hess} g(e_i, e_i) + (r+1) \cdot S_{r+1} \cdot \langle \nabla g, \eta \rangle \right] \right\}. \end{aligned}$$

Computing the Hessian of g we have that

$$\begin{aligned}
\operatorname{Hess} g\,(e_i,e_i) &= \langle \nabla_{e_i}\nabla g, e_i\rangle = -2\langle \nabla_{e_i}\rho\,\nabla\rho, e_i\rangle \\
&= -2\langle \nabla\rho, e_i\rangle^2 - 2\rho\,\langle \nabla_{e_i}\nabla\rho, e_i\rangle \qquad (3.11)\\
&= -2\langle \nabla\rho, e_i\rangle^2 - 2\rho \operatorname{Hess}\rho\,(e_i,e_i).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
-\frac{L_r f}{f} &= \frac{2}{R^2-\rho^2}\Bigg[\sum_{i=1}^n \mu_i^r\,[\langle \nabla\rho, e_i\rangle^2 + \rho \operatorname{Hess}\rho\,(e_i,e_i)] \\
&\quad + (r+1)\cdot S_{r+1}\cdot\rho\cdot\langle\nabla\rho,\eta\rangle\Bigg]
\end{aligned}$$

Setting $e_i^T = \langle\nabla\rho, e_i\rangle\nabla\rho$ and $e_i^\perp = e_i - e_i^T$, by the Hessian Comparison Theorem we have that

$$\sum_{i=1}^n \mu_i^r[\langle\nabla\rho, e_i\rangle^2 + \rho\operatorname{Hess}\rho\,(e_i,e_i)] \geq \sum_{i=1}^n \mu_i^r\,[\,\|e_i^T\|^2 + \rho\cdot\upsilon(\rho)\|e_i^\perp\|^2] \qquad (3.12)$$

and

$$(r+1)\cdot S_{r+1}\cdot\rho\cdot\langle\nabla\rho,\eta\rangle \leq (r+1)R\cdot h_{r+1}(p,R) \qquad (3.13)$$

From (3.12) and (3.13) we have that

$$\begin{aligned}
\lambda^1(\Omega) &\geq \inf_\Omega(-L_r f/f) \\
&\geq 2\cdot\inf_\Omega\Bigg\{\frac{1}{R^2-\rho^2}\Bigg[\sum_{i=1}^n \mu_i^r\,[\,\|e_i^T\|^2 + \rho\cdot\upsilon(\rho)\|e_i^\perp\|^2] \\
&\quad - (r+1)\cdot R\cdot h_{r+1}(p,R)\Bigg]\Bigg\}. \qquad (3.14)
\end{aligned}$$

If $c \leq 0$ then $\rho\cdot\upsilon(\rho) \geq 1$ thus from (3.14) we have that

$$\begin{aligned}\lambda^1(\Omega) &\geq 2\cdot\frac{1}{R^2}\left[\inf_{\Omega}\left\{\sum_{i=1}^{n}\mu_i^r\left[\|e_i^T\|^2+\|e_i^\perp\|^2\right]\right\}-(r+1)\cdot R\cdot h_{r+1}(p,R)\right]\\ &= 2\cdot\frac{1}{R^2}\left[\inf_{\Omega}\sum_{i=1}^{n}\mu_i^r-(r+1)\cdot R\cdot h_{r+1}(p,R)\right] \qquad (3.15)\\ &= 2\cdot\frac{1}{R^2}\left[(n-r)\inf_{\Omega}S_r-(r+1)\cdot R\cdot h_{r+1}(p,R)\right].\end{aligned}$$

If $c>0$ then $\rho\cdot\upsilon(\rho)=\rho\cdot\sqrt{c}\cdot\cot[\sqrt{c}\rho]\leq 1$ thus from (3.14) we have that

$$\begin{aligned}\lambda^1(\Omega) &\geq 2\cdot\frac{1}{R^2}\Bigg[\inf_{\Omega}\left\{\sum_{i=1}^{n}\mu_i^r\left[\|e_i^T\|^2+\|e_i^\perp\|^2\right]\rho\cdot\sqrt{c}\cdot\cot[\sqrt{c}\rho]\right\}\\ &\quad -(r+1)\cdot R\cdot h_{r+1}(p,R)\Bigg]\\ &= 2\cdot\frac{1}{R^2}\left[\inf_{\Omega}\left\{\sum_{i=1}^{n}\mu_i^r\rho\sqrt{c}\cot[\sqrt{c}\rho]\right\}-(r+1)\cdot R\cdot h_{r+1}(p,R)\right]\\ &= \frac{2}{R^2}\left[(n-r)\cdot R\cdot\sqrt{c}\cdot\cot[\sqrt{c}R]\cdot\inf_{\Omega}S_r-(r+1)\cdot R\cdot h_{r+1}(p,R)\right].\end{aligned}$$

To prove the Corollaries (2.1) and (2.2), observe that the hypotheses $\mu(P_r)(M)>0$ (in Corollary 2.1) and $H_{r+1}>0$ (in Corollary 2.2) imply that the L_r is elliptic. If the immersion is bounded (contained in a ball of radius R, for those choices of R) and M is closed we would have by one hand that the L_r-fundamental tone would be zero and by Theorem (2.1) that it would be positive. Then M can not be closed if the immersion is bounded. On the other hand if M is closed a ball of radius R centered at the barycenter of M could not contain M because the fundamental tone estimates for any connected component $\Omega\subset\varphi^{-1}(\varphi(M)\cap B_{\mathbb{N}^{n+1}(c)}(p,R)$ is positive. Showing that $M\neq\Omega$. The corollary (2.3) follows from item i. of Theorem (2.1) placing $S_{r+1}=0$.

3.2 Proof of Theorem 2.2

Let $\varphi : W \hookrightarrow \mathbb{N}^{n+1}(c)$ be an isometric immersion of an oriented n-dimensional Riemannian manifold W into a $(n+1)$-dimensional simply connected space form of sectional curvature c. Let $M \subset W$ be a domain with compact closure and piecewise smooth non-empty boundary and suppose that the Newton operators P_r and P_s, $0 \leq s, r \leq n-1$ are positive definite when restricted to M. Let $\mu(r) = \mu(P_r, M)$, $\mu(s) = \mu(P_s, M)$ and $\nu(r) = \nu(P_r, M)$, $\nu(s) = \nu(P_s, M)$. Given a vector field X on M we can find a vector field Y on M such that $P_r X = \kappa \cdot P_s Y$, κ constant. Now

$$
\begin{aligned}
\operatorname{div}(P_r X) - |P_r^{1/2} X|^2 &= \kappa \cdot \operatorname{div}(P_s Y) - \langle P_r X, X\rangle \\
&= \kappa \cdot \operatorname{div}(P_s Y) - \kappa^2 \langle P_s Y, P_r^{-1} P_s Y\rangle \qquad (3.16)\\
&= \kappa \cdot \left[\operatorname{div}(P_s Y) - |P_s^{1/2} Y|^2 + |P_s^{1/2} Y|^2 - \kappa \cdot |P_r^{-1/2} P_s Y|^2\right]
\end{aligned}
$$

Consider $\{e_i\}$ be an orthonormal basis such that $P_r e_i = \mu_i^r e_i$ and $P_s e_i = \mu_i^s e_i$. Letting $Y = \sum_{i=1}^n y_i e_i$ then

$$
\begin{aligned}
|P_s^{1/2} Y|^2 - \kappa \cdot |P_r^{-1/2} P_s Y|^2 &= \sum_{i=1}^n \mu_i^s y_i^2 - \kappa \sum_{i=1}^n \frac{(\mu_i^s)^2}{\mu_i^r} y_i^2 \\
&= \sum_{i=1}^n \mu_i^s y_i^2 \left[1 - \kappa \cdot \frac{\mu_i^s}{\mu_i^r}\right] \qquad (3.17)\\
&\geq 0, \ if \ \kappa \leq \frac{\mu(r)}{\nu(s)}
\end{aligned}
$$

Combining (3.16) with (3.17) and by Theorem (1.1) we have that

$$
\begin{aligned}
\lambda^{L_r}(M) &= \sup_X \inf_M \operatorname{div}(P_r X) - |P_r^{1/2} X|^2 \\
&\geq \kappa \cdot \sup_Y \inf_M \operatorname{div}(P_s Y) - |P_s^{1/2} Y|^2 = \kappa \cdot \lambda^{L_s}(M),
\end{aligned}
$$

for every $0 < \kappa \leq \dfrac{\mu(r)}{\nu(s)}$. This proves (2.6).

3.3 Proof of Theorem 2.3

Recall that for any smooth symmetric section $\Phi : \Omega \to \mathrm{End}(T\Omega)$ there is an open and dense subset $U \subset \Omega$ where the ordered eigenvalues $\{\mu_1(x) \leq \ldots \leq \mu_n(x)\}$ of $\Phi(x)$ depend smoothly on $x \in U$ and continuously in all Ω. In addition, the respective eigenvectors $\{e_1(x), \ldots, e_n(x)\}$ form a smooth orthonormal frame in a neighborhood of every point of U, see [23]. Let $f \in C_0^2(\Omega) \setminus \{0\}$ ($f \in C^2(\Omega)$ with $\int_\Omega f = 0$) be an admissible function for (the closed L_Φ-eigenvalue problem if Ω is a closed manifold) the Dirichlet L_Φ-eigenvalue problem. It is clear that f is an admissible function for the respective $\triangle$-eigenvalue problem. Writing $\nabla f(x) = \sum_{i=1}^n e_i(f) e_i(x)$ we have that

$$\begin{aligned} |\Phi^{1/2}\nabla f|^2(x) &= \langle \Phi \nabla f, \nabla f\rangle(x) \\ &= \langle \sum_{i=1}^n \mu_i(x) e_i(f) e_i, \sum_{i=1}^n e_i(f) e_i\rangle \\ &= \sum_{i=1}^n \mu_i(x) e_i(f)^2(x). \end{aligned} \tag{3.18}$$

From (3.18) we have that

$$\nu(\Phi, M)\cdot|\nabla f|^2(x) \geq |\Phi^{1/2}\nabla f|^2(x) \geq \mu(\Phi, M)\cdot |\nabla f|^2(x) \tag{3.19}$$

and

$$\nu(\Phi, M)\cdot \frac{\int_M |\nabla f|^2}{\int_M f^2} \geq \frac{\int_M |\Phi^{1/2}\nabla f|^2}{\int_M f^2} \geq \mu(\Phi, M)\cdot \frac{\int_M |\nabla f|^2}{\int_M f^2} \tag{3.20}$$

Taking the infimum over all admissible functions in (3.20) we obtain (2.8).

4 Proof of Theorem 1.2

Following [8] and [9] we shall introduce geometric invariants associated to certain spaces of vector fields that will be used to give lower bounds for the fundamental tone for p-laplacian. In this direction we begin with

Definition 4.1. *Let $\Omega \subset M$ be open subset of a smooth Riemannian manifold M. Let $\mathfrak{X}(\Omega)$ be the set of all smooth vector fields, X, on Ω with sup norm $\|X\|_\infty = \sup_\Omega |X| < \infty$ and $\inf_\Omega \operatorname{div} X > 0$. Define $c(\Omega)$ by*

$$c(\Omega) := \sup\left\{\frac{\inf_\Omega \operatorname{div} X}{\|X\|_\infty};\ X \in \mathfrak{X}(\Omega)\right\} \tag{4.1}$$

Let $X \in \mathfrak{X}(\Omega)$ a smooth vector field and $f \in \mathcal{C}_0^\infty(\Omega)$ any positive function, then the vector field f^pX has compact support in Ω. We compute the divergence of f^pX

$$\begin{aligned} 0 = \int_\Omega \operatorname{div}(f^pX)\, d\nu &= \int_\Omega [\langle \nabla(f^p), X\rangle + f^p \operatorname{div}(X)]\, d\nu \\ &= \int_\Omega \left[pf^{p-1}\langle \nabla f, X\rangle + f^p \operatorname{div}(X)\right] \nu \\ &\geq \int_\Omega \left[-p|f|^{p-1}|\nabla f||X| + f^p \operatorname{div}(X)\right] \nu \end{aligned}$$

by the Cauchy-Schwartz inequality. That is

$$0 \geq \int_\Omega \left\{-p|f|^{p-1}|\nabla f||X| + f^p \operatorname{div}(X)\right\} \nu \tag{4.2}$$

The Young inequality states that for any α, $\beta > 0$

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \qquad \text{if} \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.3}$$

It implies that for any $\varepsilon > 0$

$$\alpha\beta \leq \frac{\alpha^p}{p\,\varepsilon^p} + \frac{\varepsilon^q \beta^q}{q}. \tag{4.4}$$

Apply the Young inequality (4.4) to the inequality (4.2), letting

$$\alpha := p|\nabla f| \qquad \text{and} \qquad \beta := |f|^{p-1}|X|$$

to get

$$\begin{aligned}
0 \quad &\geq \quad \int_\Omega \left\{ -\frac{(p|\nabla f|)^p}{p\varepsilon^p} - \frac{\varepsilon^q(|f|^{p-1}|X|)^q}{q} + f^p \operatorname{div}(X) \right\} \nu \\
&= \quad \int_\Omega \left\{ -\frac{p^{p-1}}{\varepsilon^p}|\nabla f|^p - \frac{\varepsilon^q|X|^q}{q}|f|^{(p-1)q} + f^p \operatorname{div}(X) \right\} \nu \\
&= \quad \int_\Omega \left\{ -\frac{p^{p-1}}{\varepsilon^p}|\nabla f|^p + \Big(\operatorname{div}(X) - \frac{\varepsilon^q|X|^q}{q}\Big)|f|^p \right\} \nu \\
&\geq \quad -\frac{p^{p-1}}{\varepsilon^p}\int_\Omega |\nabla f|^p\, \nu + \Big(\inf_\Omega \operatorname{div}(X) - \frac{\varepsilon^q}{q}\sup_\Omega |X|^q\Big)\int_\Omega |f|^p\, \nu
\end{aligned}$$

that is

$$\frac{p^{p-1}}{\varepsilon^p}\int_\Omega |\nabla f|^p\, \nu \geq \Big(\inf_\Omega \operatorname{div}(X) - \frac{\varepsilon^q}{q}\sup_\Omega |X|^q\Big)\int_\Omega |f|^p\, \nu \tag{4.5}$$

or else

$$\int_\Omega |\nabla f|^p\, \nu \quad \geq \quad \frac{\varepsilon^p}{p^{p-1}}\Big(\inf_\Omega \operatorname{div}(X) - \frac{\varepsilon^q}{q}\sup_\Omega |X|^q\Big)\int_\Omega |f|^p\, \nu \tag{4.6}$$

When $\operatorname{div}(X) \leq 0$ on Ω, the previous inequality is trivial and does not bring any interesting information. So we shall assume tacitly that $\operatorname{div}(X) \geq 0$ on Ω. Consider the function

$$\psi(\varepsilon) = \varepsilon^p (A - B\varepsilon^q)$$

with $A \geq 0$ and $B > 0$. We will look for the maximum this function assumes as a function of A and B. This is a straightforward calculation:

- $\psi'(\varepsilon) = \varepsilon^{p-1}[pA - (p+q)B\varepsilon^q]$.
- thus the zeroes of ψ' are given by

$$\varepsilon_1 = 0 \qquad \text{and} \qquad \varepsilon_2 = \left(\frac{pA}{(p+q)B}\right)^{1/q}.$$

- $\psi''(\varepsilon) = \varepsilon^{p-2}[p(p-1)A - (p+q)(p+q-1)B\varepsilon^q]$.
- calculating ψ'' on both ε_1 and ε_2 we get

$$\psi''(\varepsilon_1) = 0 \qquad \text{and} \qquad \psi''(\varepsilon_2) = -pq\varepsilon_2^{p-2}A \leq 0.$$

- consequently ε_2 is a maximum and the maximum value of ψ is given by

$$\psi(\varepsilon_2) = \left(\frac{pA}{(p+q)B}\right)^{p/q} \frac{qA}{p+q} = \frac{qp^{p/q}A^p}{(p+q)^p B^{p/q}},$$

  since $1 + p/q = p$.

Letting $A = \inf_\Omega \operatorname{div}(X)$ and $B = \sup_\Omega |X|^q/q$ we observe that

$$\max_\varepsilon \left[\varepsilon^p\left(\inf_\Omega \operatorname{div}(X) - \frac{\varepsilon^q \sup_\Omega |X|^q}{q}\right)\right] = \frac{q^p p^{p/q}}{(p+q)^p} \frac{(\inf_\Omega \operatorname{div}(X))^p}{\sup_\Omega |X|^p}$$

and consequently

$$\frac{1}{p^{p-1}} \max_\varepsilon \left[\varepsilon^p\left(\inf_\Omega \operatorname{div}(X) - \frac{\varepsilon^q \sup_\Omega |X|^q}{q}\right)\right] = \frac{1}{p^p}\left(\frac{\inf_\Omega \operatorname{div}(X)}{\sup_\Omega |X|}\right)^p \quad (4.7)$$

inserting the estimate (4.7) in (4.6) we get

$$\begin{aligned}\int_\Omega \|\nabla f|^p\,\nu &\geq \frac{1}{p^p}\left(\frac{\inf_\Omega \mathrm{div}(X)}{|X|_\infty}\right)^p \int_\Omega |f|^p\,\nu\\ &\geq \frac{1}{p^p}\left(\sup_{X\in\mathfrak{X}(\Omega)} \frac{\inf_\Omega \mathrm{div}(X)}{|X|_\infty}\right)^p \int_\Omega |f|^p\,\nu\end{aligned}$$

and thus

$$\int_\Omega |\nabla f|^p\,\nu \geq \frac{c(\Omega)^p}{p^p}\int_\Omega |f|^p\,\nu \tag{4.8}$$

leading to the estimate for the fundamental tone

$$\mu_p^*(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla f|^p}{\int_\Omega |f|^p} : f\in W_+^{1,p}(\Omega)\setminus\{0\}\right\} \geq \frac{c(\Omega)^p}{p^p} \tag{4.9}$$

This concludes the proof of Theorem 1.2.

To prove McKean's generalized Theorem 1.1 we take $X = \nabla\rho$, the gradient of distance function from a point o and observe that $|\nabla\rho| = 1$. On the other hand $\mathrm{div}(\nabla\rho) = \triangle\rho$. Now, since $K_M \leq -c^2 < 0$ the laplacian comparison theorem implies that $\triangle\rho \geq (n-1)c$. Hence

$$\frac{(n-1)^p c^p}{p^p} \leq \frac{1}{p^p}\left(\frac{\mathrm{div}(\nabla\rho)}{|\nabla\rho|}\right)^p \leq \frac{c(M)^p}{p^p} \leq \lambda_p^*(M)$$

concluding the proof.

4.1 Geometric application of Theorem 1.2

Let $B(o,r)$ be the geodesic ball of radius r and center $o\in M$ of a Riemannian n-manifold. Suppose that the radius satisfies

$$r < \min\{\mathrm{inj}(o), \pi/2\kappa(r)\},$$

where $\kappa(r)$ is the supremum of the sectional curvatures $K(\sigma)$ of all two dimensional planes $\sigma \subset T_xM$, $x \in B(o,r)$,

$$\kappa(r) = \sup_{B(o,r)} \{K(\sigma), \sigma \subset T_xM, x \in B(o,r)\}.$$

If $\kappa(r) \leq 0$ we assume that $\pi/2\kappa(r) = \infty$. Let $\rho\colon M \to \mathbb{R}$ be the distance function to the point $o \in M$. Set $f\colon M \to \mathbb{R}$ given by $f(x) = \rho^q(x) = \rho(x)^{p/(p-1)}$ and $X = |\nabla f|^{p-2}\nabla f$. Since $\nabla f = q\rho^{q-1}\nabla\rho$ it follows that

$$\begin{aligned} X &= (q\rho^{q-1})^{p-2}q\rho^{q-1}\nabla\rho = q^{(p-2)+1}\rho^{(q-1)(p-2)+(q-1)}\nabla\rho \\ &= q^{p-1}\rho^{(q-1)(p-1)}\nabla\rho = q^{p-1}\rho\nabla\rho \end{aligned}$$

thus $|X| \leq q^{p-1}R$ on the ball $B_R(o)$. Computing the divergence of X we have

$$\begin{aligned} \mathrm{div}X &= \mathrm{div}(q^{p-1}\rho\nabla\rho) = q^{p-1}\Big(\langle\nabla\rho,\nabla\rho\rangle + \rho\triangle\rho\Big) \\ &= q^{p-1}\Big(1+\rho\triangle\rho\Big) \\ &\geq q^{p-1}\Big(1+(n-1)\rho(x)\frac{h'}{h}(\rho(x))\Big) \end{aligned}$$

where h is the solution of the Cauchy problem $h''(t)+\kappa(t)h(t) = 0$ for all $t \in [0,r]$ with initial conditions $h(0) = 0$, $h'(0) = 1$.

Thus

$$\begin{aligned} \lambda_p^*(B_R(o)) &\geq \left(\frac{\inf_{B_R(o)}\mathrm{div}(X)}{p\|X\|_\infty}\right)^p \\ &\geq \left(\frac{(1+(n-1)\inf_{B_R(o)}\rho(x)(h'/h)(\rho(x)))}{pR}\right)^p. \end{aligned}$$

Summarizing the discussion, we get the following generalization of Theorem (4.1) of [8]

Theorem 4.1. *Let M a n-dimensional Riemannian manifold and $B(o,r)$ a geodesic ball with radius $r < \min\{inj(o), \pi/2\kappa(r)\}$. Then the p-fundamental tone of $B(0,r)$ has lower bound given by*

$$\begin{aligned}\lambda_p^*(B_R(o)) &\geq \left(\frac{\inf_{B_R(o)} \operatorname{div}(X)}{p\|X\|_\infty}\right)^p \\ &\geq \left(\frac{(1+(n-1)\inf_{B_R(o)} \rho(x)(h'/h)(\rho(x)))}{pR}\right)^p.\end{aligned}$$

5 Proof of Theorem 1.3

Let $X \in \mathcal{W}^{1,1}(M)$ and $f \in C_0^\infty(M)$.

$$\begin{aligned}0 &= \int_M \operatorname{div}(f^p X)\, d\nu = \int_M \Big[\langle \nabla(f^p), X\rangle + f^p \operatorname{div}(X)\Big]\, d\nu \\ &= \int_M \Big[p f^{p-1}\langle \nabla f, X\rangle + f^p \operatorname{div}(X)\Big]\, d\nu \\ &\geq \int_M \Big[-p f^{p-1}|\nabla f||X| + f^p \operatorname{div}(X)\Big]\, d\nu \\ &\geq \int_M \left[-p\Big(\frac{|\nabla f|^p}{p} + \frac{f^{(p-1)q}|X|^q}{q}\Big) + f^p \operatorname{div}(X)\right] d\nu \\ &\geq -\int_M |\nabla f|^p\, d\nu + \int_M \left(-\frac{p}{q}|X|^q + \operatorname{div}(X)\right) f^p\, d\nu \\ &\geq -\int_M |\nabla f|^p\, d\nu + \inf_M \Big((1-p)|X|^q + \operatorname{div}(X)\Big) \int_M f^p\, d\nu\end{aligned}$$

where we have used the Young inequality $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ for the pair $a = |\nabla f|$ and $b = f^{p-1}|X|$ and the fact that the exponents p,q are conjugate, that is $(p-1)q = p$. Thus we have

$$\int_M |\nabla f|^p \, d\nu \geq \inf_M \Big((1-p)|X|^q + \operatorname{div}(X) \Big) \int_M f^p \, d\nu$$

or

$$\frac{\int_M |\nabla f|^p \, d\nu}{\int_M f^p \, d\nu} \geq \inf_M \Big((1-p)|X|^q + \operatorname{div}(X) \Big)$$

for any vector field $X \in \mathcal{W}^{1,1}(M)$, hence we obtain

$$\frac{\int_M |\nabla f|^p \, d\nu}{\int_M f^p \, d\nu} \geq \sup_{X \in \mathcal{W}^{1,1}(M)} \inf_M \Big((1-p)|X|^q + \operatorname{div}(X) \Big) \tag{5.1}$$

Thus

$$\lambda_p^*(M) = \inf_{W_0^{1,p}(M)} \frac{\int_M |\nabla f|^p \, d\nu}{\int_M |f|^p \, d\nu} \geq \sup_{X \in \mathcal{W}^{1,1}(M)} \inf_M \Big((1-p)|X|^q + \operatorname{div}(X) \Big) \tag{5.2}$$

Let u be an eigenfunction associated to the least eigenvalue $\lambda_{1,p}$, that is

$$\triangle_p u = \lambda_{1,p}(\Omega)|u|^{p-2}u$$

and consider the vector field

$$X = -\frac{|\nabla u|^{p-2}\nabla u}{|u|^{p-2}u} \tag{5.3}$$

calculating its norm and divergence we obtain

$$\begin{aligned} |X|^q &= \left(\frac{|\nabla u|^{p-2}}{|u|^{p-1}} |\nabla u| \right)^q \\ &= \left(\frac{|\nabla u|^{p-1}}{|u|^{p-1}} \right)^q = \frac{|\nabla u|^{(p-1)q}}{|u|^{(p-1)q}} = \frac{|\nabla u|^p}{|u|^p} \end{aligned}$$

and

$$\begin{aligned}
\operatorname{div}(X) &= -\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{u^{p-1}}\right)\\
&= -\frac{\operatorname{div}(|\nabla u|^{p-2}\nabla u)}{u^{p-1}} - \langle |\nabla u|^{p-2}\nabla u, \nabla \frac{1}{u^{(p-1)}}\rangle\\
&= \frac{\triangle_p u}{|u|^{p-2}u} + (p-1)u^{-p}|\nabla u|^{p-2}\langle \nabla u, \nabla u\rangle\\
&= \mu_{1,p}(\Omega) + (p-1)\frac{|\nabla u|^p}{u^p}.
\end{aligned}$$

Gathering these results

$$\begin{aligned}
(1-p)|X|^q + \operatorname{div}(X) &= (1-p)\frac{|\nabla u|^p}{|u|^p} + \mu_{1,p}(\Omega) + (p-1)\frac{|\nabla u|^p}{u^p}\\
&= \lambda_{1,p}(\Omega).
\end{aligned}$$

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A Note on the Maximum Principle at Infinity

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In honor of Professor Barnabé Pessoal Lima on the occasion of his 60[th] birthday.

Abstract: In this short note we present a simple proof of the Omori-Yau maximum principle, and show two applications of this important maximum principle at infinity for unbounded isometric immersions into product spaces. Specifically, we prove sectional curvature estimates for submanifolds immersed into a Riemannian warped product space, and a slice type theorem for spacelike hypersurfaces with positive constant higher order mean curvature immersed in a Robertson-Walker space.

1 The maximum principle at infinity

Omori [23], studying isometric immersions of minimal submanifolds into cones of $\mathbb{R}^n$, introduce the following global version of the maximum principle for complete Riemannian manifolds satisfying some constraints on the sectional curvature.

Theorem 1.1 (Omori [23]). *Let M be a complete Riemannian manifold and suppose that the sectional curvature is bounded below by $K_M \geq -\Lambda^2$. If $u \in C^2(M)$ with $u^* = \sup_M u < \infty$ then there*

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exists a sequence of points $x_k \in M$ *such that*

$$\lim_{k\to\infty} u(x_k) = u^*, \quad |\nabla u|(x_k) < \frac{1}{k}, \quad \mathrm{Hess}\, u(x_k)(X,X) < \frac{1}{k}\cdot |X|^2,$$

for every $X \in T_{x_k}M$.

Omori's maximum principle was refined and extended by Yau in a series of papers [13], [34] and [35] to Riemannian manifolds with Ricci curvature bounded below and applied to find elegant solutions to various analytic-geometric problems on Riemannian manifolds. Pigola, Rigoli and Setti [28] introduced the following terminology.

Definition 1.1. *The Omori-Yau maximum principle for the Laplacian is said to hold on M if for any given* $u \in C^2(M)$ *with* $u^* < \infty$, *there exists a sequence of points* $x_k \in M$ *such that*

$$\lim_{k\to+\infty} u(x_k) = u^*, \quad |\nabla u|(x_k) < \frac{1}{k}, \quad \triangle u(x_k) < \frac{1}{k}.$$

Likewise, the Omori-Yau maximum principle for the Hessian is said to hold on M if for any given $u \in C^2(M)$ *with* $u^* < \infty$, *there exists a sequence of points* $x_k \in M$ *such that*

$$\lim_{k\to\infty} u(x_k) = u^*, \quad |\nabla u|(x_k) < \frac{1}{k}, \quad \mathrm{Hess}\, u(x_k)(X,X) < \frac{1}{k}\cdot |X|^2,$$

for every $X \in T_{x_k}M$.

It is well-known that the Hessian and Laplacian Omori-Yau maximum principles depend in a subtle way on the geometry of a complete manifold M (see for instance [12, 31]). A systematic study has been undertaken for various authors [28], [27, 9], [7], where the authors described a general function-theoretic criterion.

Theorem 1.2. *The Omori-Yau maximum principle holds for the Laplacian provided that M supports a function $0 < \gamma \in C^2(M\backslash K)$, for some compact K, with the following properties:*

i. $\gamma(x) \to +\infty$ *as* x *diverges*,

ii. $|\nabla\gamma| \le F(\gamma)$,

iii. $\Delta\gamma \le F(\gamma)$,

for some F satisfying

$$0 < F \in C^1(\mathbb{R}^+), \qquad F' \ge 0, \qquad F^{-1} \notin L^1(+\infty). \tag{1.1}$$

For the Hessian principle, the last condition in (1.2) *has to be replaced by* $\nabla \mathrm{d}\gamma \le F(\gamma)\langle\, ,\rangle$.

In a nomenclature due to [2], the pair (γ, F) is called an Omori-Yau pair for M. The above criterion described in Theorem 1.2 is called the Khas'minskii test, and it is effective since the Khas'minskii potential γ in Theorem 1.2 can be explicitly found in a number of geometrically relevant applications (see [28, 7]): for instance, letting $\gamma(x) = \rho(x)$ denote the distance from a fixed origin o, when M is complete the Khas'minskii test holds if the Ricci curvature satisfies

$$\mathrm{Ric}_x(\nabla\rho, \nabla\rho) \ge -F^2\big(\rho(x)\big) \quad \text{outside of } \mathrm{cut}(o). \tag{1.2}$$

However, γ might be independent of the curvatures of M, and actually there are cases when (1.2) is met but the sectional curvature of M goes very fast to $-\infty$ along some sequence [28, p. 13]. For the Hessian principle, (1.2) has to be replaced by an analogous decay for all of the sectional curvatures of M:

$$K_M(\pi_x) \geq -F^2\big(\rho(x)\big) \qquad \begin{array}{l} \forall\, x \notin \mathrm{cut}(o), \\ \forall\, \pi_x \leq T_xM \;\; \text{2-plane containing } \nabla\rho. \end{array} \tag{1.3}$$

Note that (1.1) includes the case when $F(t)$ is constant, considered in [23, 34]. With simple manipulation, we can extract from Theorem 1.2 an easy form for the Khas'minskii test: first, we observe that up to replacing γ with

$$\int_0^\gamma \frac{\mathrm{d}s}{F(s)},$$

without loss of generality we can choose $F(\gamma) \equiv 1$; next, since γ is an exhaustion, for $\lambda > 0$ fixed it holds $\Delta\gamma \leq 1 \leq \lambda\gamma$ on $M \backslash K$, if K is large enough. In other words, from the conditions *i.*, *ii.* and *iii.* in Theorem 1.2 we can produce a new γ satisfying $|\nabla\gamma| \leq 1$, $\Delta\gamma \leq \lambda\gamma$ outside some compact set.

Proof of Theorem 1.2:
We fix a sequence of positive real numbers $(\varepsilon_k)_{k\in\mathbb{N}}$ such that, $\varepsilon_k \to 0$ and consider now any function $u \in C^2(M)$ bounded from above. Define $g_k(x) = u(x) - \varepsilon_k\varphi(\gamma(x))$, where

$$\varphi(t) = \int_0^t \frac{ds}{F(s)}. \tag{1.4}$$

Observe that φ is $C^2(M)$, positive and satisfies $\varphi(t) \to +\infty$ as $t \to +\infty$. By a direct computation we have

$$\begin{aligned} \varphi'(t) &= \frac{1}{F(t)}, \\ \varphi''(t) &= -\frac{F'(t)}{F^2(t)}, \end{aligned}$$

and using the properties satisfied by F we conclude that

$$\varphi''(t) \le 0. \tag{1.5}$$

It is clear that g_k attains its supremum at some point $x_k \in M$. This gives the desired sequence x_k. It follows directly from definition of g_k that

$$\nabla g_k(x) = \nabla u(x) - \varepsilon_k \varphi'(\gamma(x))\nabla\gamma(x).$$

In particular, at the points x_k, using (1.2), we obtain

$$|\nabla u|(x_k) = \varepsilon_k \varphi'(\gamma(x_k))|\nabla\gamma|(x_k) \le \varepsilon_k. \tag{1.6}$$

Computing $\mathrm{Hess}\, g_k(x)(v,v)$ we have

$$\begin{aligned}
\mathrm{Hess}\, g_k(x)(X,X) &= \mathrm{Hess}\, u(x)(X,X) - \varepsilon_k \varphi'(\gamma(x))\mathrm{Hess}\,\gamma(x)(X,X) \\
&\quad - \varepsilon_k \varphi''(\gamma(x))\langle \nabla\gamma(x), X\rangle^2 \qquad (1.7)\\
&\ge \mathrm{Hess}\, u(x)(X,X) - \varepsilon_k \varphi'(\gamma(x))\mathrm{Hess}\,\gamma(x)(X,X)
\end{aligned}$$

for all $X \in T_xM$. Again, since x_k is a maximum point of g_k, by (1.2) and the expression of φ', we get

$$\mathrm{Hess}\, u(x_k)(X,X) \le \varepsilon_k \varphi'(\gamma(x_k))\mathrm{Hess}\,\gamma(x_k)(X,X) \le \varepsilon_k \langle X,X\rangle. \tag{1.8}$$

Finally, in the case of the Laplacian, we obtain

$$\Delta u(x_k) \le \varepsilon_k \varphi'(\gamma(x_k))\Delta\gamma(x_k) \le \varepsilon_k.$$

To finish the proof we need to show that $u(x_k) \to u^*$. To do that, we follow [28] closely and observe that for any fixed $j \in \mathbb{N}$, there is a $y \in M$ such that $u(y) > \sup u - 1/2j$.

Since g_k has a maximum at x_k we have

$$u(x_k) - \varepsilon_k \varphi(\gamma(x_k)) = g_k(x_k) \geq g_k(y) = u(y) - \varepsilon_k \varphi(\gamma(y)).$$

Therefore

$$u(x_k) > \sup u - \frac{1}{2j} - \varepsilon_k \varphi(\gamma(y)). \tag{1.9}$$

Choosing $k = k(j) = k_j$ sufficiently large such that

$$\varepsilon_{k_j} \varphi(\gamma(y)) < \frac{1}{2j}, \tag{1.10}$$

it follows from (1.9) and (1.10) that

$$u(x_{k_j}) > \sup u - \frac{1}{2j} - \frac{1}{2j} = \sup u - \frac{1}{j}. \tag{1.11}$$

Therefore $\lim_{j \to +\infty} u(x_{k_j}) = u^*$ and this finishes the proof.

Remark 10. *It is worth to remark that the above proof holds true for smooth functions u satisfying the growth*

$$\lim_{x \to \infty} \frac{u(x)}{\varphi(\gamma(x))} = 0,$$

where φ was defined in (1.4)*. This allows to consider unbounded functions, which enlarge the range of applications of the Omori-Yau maximum principle (cf. [9, 10, 11]).*

The Omori-Yau maximum principle can be also considered for differential elliptic operators other than the Laplacian (see [7] for a complete overview), like the ϕ-Laplacian, see [29] and semi-elliptic trace operators $L = Tr(P \circ \text{hess})$ considered in [20], [8] and [6], where $P: TM \to TM$ is a positive semi-definite symmetric tensor on TM and for each $u \in C^2(M)$, $\mathrm{hess}\, u: TM \to TM$ is a symmetric operator defined by $\mathrm{hess}\, u(X) = \nabla_X \nabla u$ for every $X \in TM$. Here, we are going to consider the trace operator

$$Lu = Tr(P \circ \mathrm{hess}\, u) + \langle V, \nabla u \rangle \tag{1.12}$$

with $\sup_M |V| < +\infty$, and state an Omori-Yau maximum principle in the same spirit of Theorem 1.9 in [28], see also [18] and [1]. The following result is contained in [10] and it is a slight extension of Theorem 1 of [6]. Its proof is very similar to that Theorem 1.2.

Theorem 1.3. *Let M be a complete Riemannian manifold. Consider a semi-elliptic operator L defined as in* (1.12)*, where V is bounded. Suppose that there exists a Khas'minskii potential* $0 < \gamma \in C^2(M \backslash K)$*, for some compact K, with the following properties:*

i. $\gamma(x) \to +\infty$ *as* x *diverges*,

ii. $|\nabla \gamma| \leq F(\gamma)$,

iii. $\mathrm{tr}\,(\mathrm{P} \circ \mathrm{hess}\, \gamma) \leq F(\gamma)$,

where F satisfies (1.1)*. Then given any function $u \in C^2(M)$ satisfying* (1.4)*, there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset M$ satisfying:*

$$(a)\ \ |\nabla u|(x_k) < \frac{1}{k} \qquad \text{and} \qquad (b)\ \ Lu(x_k) < \frac{1}{k}.$$

If we suppose that u is bounded above we have that

$$(c) \lim_{k \to +\infty} u(x_k) = u^*.$$

Corollary 1.1. *Let $(M, \langle , \rangle)$ be a complete, non-compact, Riemannian manifold with radial sectional curvature satisfying*

$$K_M(\pi_x) \geq -B^2\big(\rho(x)\big)^2 \qquad \begin{array}{l} \forall\, x \notin \mathrm{cut}(o), \textit{ outside a compact set} \\ \forall\, \pi_x \leq T_x M \;\; \textit{2-plane containing } \nabla\rho. \end{array}$$

for some constant $B \in \mathbb{R}_+$. *Then, the Omori-Yau maximum principle for the semi-elliptic operator* L *defined in* (1.12) *holds on* M *provided* V *is bounded and* $\operatorname{tr} \mathrm{P} = o(\log r)$ *as* $r \to +\infty$.

Proof. Following the same steps of the Example 1.13 in [28] one has that above bound on the sectional curvature gives rise to

$$\operatorname{Hess} r \leq Dr,$$

with $D > 0$. Then, taking $\gamma = r$ and $F(t) = Dt \log t$, we conclude that

$$\operatorname{tr}(P \circ \operatorname{hess}) \leq F(\gamma).$$

Therefore, we conclude from Theorem 1.3. □

2 Geometric Applications

The main goal of this section is describe two ways to obtain nice geometric resuts basing on ideas from the maximum principle at infinity. In one hand, we derive curvature estimates of submanifolds immersed on a warped product manifold by making use of the method implemented to prove Theorem 1.2. On the other hand, as a direct application of the Omori-Yau maximum principle we prove slice theorems for submanifolds immersed on Roberson-Walker product space. In both applications the submanifolds considered could be unbounded.

2.1 Curvature Estimates

Since the estimates that will be given depends on the warping function and on the geometry of the manifolds involved, it is convenient to present briefly a basic introduction on the warped product. We will follow closely the book of [25].

We know that on the product manifold $L \times N$ the metric tensor is given by $\pi_L^*(g_L) + \pi_N^*(g_N)$, where π_L and π_N are the canonical projections of $L \times N$ onto L and N, respectively. Letting $\rho : L \to \mathbb{R}_+$ be a positive smooth function, we define the warped product $L \times_\rho N$ by the product manifold $L \times N$ furnished with metric tensor

$$g = \pi_L^*(g_L) + (\rho \circ \pi_L)^2 \pi_N^*(g_N). \tag{2.1}$$

Writing $P = L \times_\rho N$, the goal is to express the geometry of P in terms of warping function ρ and the geometries of L and N. L is called the base of P and N the fiber. The relation of a warped product to the base L is almost as simple as in the trivial case of a Semi-Riemannian product. Although, the relation to the fiber N often involves the warping function ρ. In the next lemma we collect some important properties.

Lemma 2.1. *Let $f : L \to \mathbb{R}$ be a smooth function and $T, S \in TL$ and $X, Y \in TN$. The following relations holds on $P = L \times_\rho N$:*

a) $d\pi_L \nabla^P (f \circ \pi_L) = \nabla^L f$,

b) $\nabla_S^P T = \nabla_S^L T$,

c) $\nabla_X^P T = \nabla_T^P X = T(\eta) X$,

d) $\nabla_X^P Y = \nabla_X^N Y - \langle X, Y \rangle_P \nabla^L \eta$,

where $\eta = \ln \rho$.

ϕ_h-Bounded Submanifolds $L \times_\rho N$

Let N be a geodesically complete Riemannian manifold with a distinguished point z_0 and radial sectional curvature along the geodesics issuing from z_0,

$$K_N^{\mathrm{rad}}(z) \le -G(\rho_N(z)) \le b \le 0 \tag{2.2}$$

where $\rho_N(z) = \mathrm{dist}_N(z_0, z)$ and $G \in C^\infty([0,\infty))$. Let h be the solution of the Cauchy problem

$$\begin{cases} h''(t) - G(t)h(t) = 0, \\ h(0) = 0, \quad h'(0) = 1. \end{cases} \tag{2.3}$$

Since $-G \le 0$ we have that

$$h(t) > 0 \text{ in } \mathbb{R}^+ = (0,\infty) \text{ and } \liminf_{t\to+\infty} h(t) > 0. \tag{2.4}$$

Let L be a geodesically complete ℓ-dimensional Riemannian manifold with a distinguished point y_0 and radial sectional curvature

$$K_L^{\mathrm{rad}}(y) \ge -F^2(\rho_L(y)) \tag{2.5}$$

where $\rho_L(y) = \mathrm{dist}_L(y_0, y)$, $F \in C^1(\mathbb{R}_0^+)$, $F^{-1} \notin L^1(+\infty)$, $F(0) = 1$, and $F'(t) \ge 0$. Let g be the solution of

$$\begin{cases} g''(t) - F^2(t)g(t) = 0, \\ g(0) = 0, \quad g'(0) = 1. \end{cases}$$

It follows from $F > 0$ that

$$g(t) > 0 \text{ on } \mathbb{R}^+ = (0,+\infty), \quad g'(t) \ge 0. \tag{2.6}$$

The Hessian Comparison Theorem (see [16] or [30]) applied to ρ_L yields

$$\mathrm{Hess}_L\,\rho_L \leq \frac{g'}{g}(\rho_L)\{\langle\,,\rangle_L - d\rho_L \otimes d\rho_L\}. \tag{2.7}$$

Letting

$$\psi(t) = \left(\exp\left[\int_0^t F(s)ds\right] - 1\right)$$

we see that ψ satisfies

$$\begin{cases} \psi''(t) - F^2(t)\psi(t) & \geq & 0, \\ \psi(0) = 0, \;\; \psi'(0) & = & 1. \end{cases}$$

Applying Sturm comparison theorem, see [30, Chapter 2.], we have that

$$\frac{g'}{g}(t) \leq \frac{\psi'}{\psi}(t) = F(t)\frac{\psi(t)+1}{\psi(t)}. \tag{2.8}$$

Consider the functions $\phi_h \in C^\infty(\mathbb{R}_0^+)$ and $\zeta \in C^2(\mathbb{R}_0^+)$ given by

$$\phi_h(t) = \int_0^t h(s)ds, \qquad \zeta(t) = \int_0^t \frac{ds}{F(s)}.$$

For $\varepsilon \in (0,1)$, define the region

$$\Omega_{\phi_h,\zeta}(\varepsilon) = \{(y,z) \in L \times_\rho N : \phi_h(\rho_N(z)) \leq \zeta(\rho_L(y))^{1-\varepsilon}\}.$$

Definition 2.1. *An isometric immersion $\varphi: M \to L \times_\rho N$ of a Riemannian manifold M into the product $L \times_\rho N$ is said to be ϕ_h-bounded if there exist a compact $K \subset M$ and $\varepsilon \in (0,1)$ such that $\varphi(M \backslash K) \subset \Omega_{\phi_h,\zeta}(\varepsilon)$.*

A properly immersed ϕ_h-bounded hypersurface of $L \times_\rho N$ does not need to be cylindrically bounded, since the ends of any cylinder are contained properly in $\Omega_{\phi_h,\zeta}(\varepsilon)$.

Jorge-Koutrofiotis' Estimate

The classical isometric immersion problem asks whether there exists an isometric immersion $\varphi\colon M^m \to N^n$ for given Riemannian manifolds M and N, with $m < n$. The model result for this type of problem is the celebrated Efimov-Hilbert Theorem that says that there is no isometric immersion of a geodesically complete surface M with sectional curvature $K_M \leq -\delta^2 < 0$ into $\mathbb{R}^3$, $\delta \in \mathbb{R}$, see [17] and [15]. On the other hand, the Nash Embedding Theorem shows that there is always an isometric embedding into the Euclidean n-space $\mathbb{R}^n$ provided the codimension $n-m$ is sufficiently large, see [22].

For small codimension, meaning that $n-m \leq m-1$, the answer in general depends on the geometries of M and N. For instance, a classical result of Tompkins [33] states that a compact, flat, m-dimensional Riemannian manifold can not be isometrically immersed into $\mathbb{R}^{2m-1}$ (see also [14], [26], [24], [32] and [21]). In [19] Jorge and Koutrofiotis considered this question for complete extrinsically bounded[5] submanifolds with scalar curvature bounded from below. Our first application is an extension of the Jorge-Koutrofiotis Theorem.

Theorem 2.1. *Let $\varphi\colon M \to L \times_\rho N = P$ be a properly immersed submanifold such that $\varphi(M\backslash K) \subset \Omega_{\phi_h,\zeta}(\varepsilon)$ for some compact $K \subset M$ and positive $\varepsilon \in (0,1)$. Suppose that the sectional curvatures of N and L satisfy* (2.2) *and* (2.5). *If ρ is bounded and $n-m \leq m-\ell-1$, then*

$$\sup_M \rho^{-2} K_M \geq |b| + \inf_N \rho^{-2} K_N. \tag{2.9}$$

With strict inequality $\sup_M K_M > \inf_N K_N$ if $b = 0$.

[5]Meaning: immersed into regular geodesic balls of a Riemannian manifold.

Sketch of the Proof:

Take $x_0 \in M$ such that $\varphi(x_0) = (y_0, z_0) \in P$. Define $f\colon M \to \mathbb{R}$ by $f(x) = \phi_h \circ \rho_N(z(x))$, and $p\colon M \to \mathbb{R}$ by $p(x) = \zeta \circ \rho_L(y(x))$. For each $k \in \mathbb{N}$, set $g_k(x) = f(x) - \frac{1}{k} p(x)$. Observe that $g_k(x_0) = 0$ for all k and since φ is proper $g_k(x) < 0$ whenever $\rho_M(x) \gg 1$. This implies that g_k has a maximum at a point x_k, yielding in this way a sequence $\{x_k\} \subset M$ such that $\mathrm{Hess}_M g_k(x_k) \leq 0$, in the sense of quadratic forms. Proceeding as in the proof of Theorem 1.2 we have that for $X \in T_{x_k}M$,

$$\mathrm{Hess}_M f(x_k)(X,X) \leq \frac{1}{k}\mathrm{Hess}_M p(x_k)(X,X). \tag{2.10}$$

We need to estimate both terms of (2.10). Since $\mathrm{Hess}_P p = \mathrm{Hess}_L p$ and $\zeta'' \leq 0$, for $e \in T_{x_k}M$, $|e| = 1$, we compute at x_k

$$\begin{aligned}
\mathrm{Hess}_M p(e,e) &= \mathrm{Hess}_L p(e,e) + \langle \nabla_P p, \alpha(e,e)\rangle \\
&= \zeta''(\rho_L)\langle e, \nabla_L \rho_L\rangle^2 + \zeta'(\rho_L)\mathrm{Hess}_L \rho_L(e,e) \\
&\quad + \zeta'(\rho_L)\langle \nabla_L \rho_L, \alpha(e,e)\rangle \\
&\leq \frac{1}{F(\rho_L)}\left(\mathrm{Hess}_L \rho_L(e,e) + |\alpha(e,e)|\right) \\
&\leq \frac{\psi(\rho_L)+1}{\psi(\rho_L)} + |\alpha(e,e)|, \qquad (2.11)
\end{aligned}$$

where in the last inequality we have used (2.8) and the fact that $F \geq 1$.

In order to compute the left hand side of (2.10) we set $f = \phi_h \circ g \circ \varphi$ where g is given by $g(y,z) = \rho_N(z)$. Let us consider an orthonormal basis at $T_{\varphi(x_k)}(L \times_\rho N)$

$$\{\overbrace{\partial/\partial\gamma_1, \ldots, \partial/\partial\gamma_\ell}^{\in TL}, \overbrace{\nabla\rho_N, \partial/\partial\theta_1, \ldots, \partial/\partial\theta_{n-\ell-1}}^{\in TN}\}. \tag{2.12}$$

Thus if $e \in T_{x_k}M$, $|e| = 1$, we can decompose

$$e = \sum_{i=1}^{\ell} c_i \cdot \partial/\partial\gamma_i + a \cdot \nabla\rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial/\partial\theta_j,$$

with $\rho^2\left(a^2 + \sum_{j=1}^{n-\ell-1} b_j^2\right) + \sum_{i=1}^{\ell} c_i^2 = 1$.

Applying again the Hessian Comparison Theorem (see [16] or [30]) and using that $\nabla^P g = \rho^{-2}\nabla^N\rho_N$ and

$$\mathrm{Hess}_P g(e,e) = -2\langle\nabla^L\eta, e\rangle\langle\nabla^N\rho_N, e\rangle + \mathrm{Hess}_P\rho_N(e,e),$$

we compute at x_k

$$\begin{aligned} \mathrm{Hess}_P f(e,e) &= \phi_h''(\rho_N)\langle e, \nabla^P g\rangle_P^2 + \phi_h'(\rho_N)\mathrm{Hess}_P g(e,e) \\ &\geq -2h(\rho_N)\langle\nabla^L\eta, e\rangle\langle\nabla^N\rho_N, e\rangle + h'(\rho_N)|e^N|^2, \end{aligned}$$

Recalling that $n + \ell \leq 2m - 1$. This dimensional restriction implies that for every $x \in M$ there exists a sub-space $V_x \subset T_xM$ with $\dim(V_x) \geq (m - \ell) \geq 2$ such that $V \perp TL$, this is equivalent to $c_i = 0$. If we take any $e \in V_{x_k} \subset T_{x_k}M$, $|e| = 1$ we have

$$\mathrm{Hess}_P f(e,e) \quad \geq \quad h'(\rho_N)$$

Now, we can estimate the left hand side of (2.10)

$$\begin{aligned} \mathrm{Hess}_M f(x_k)(e,e) &= \mathrm{Hess}_P f(\varphi(x_k))(e,e) + \langle\nabla_P f, \alpha_{x_k}(e,e)\rangle \\ &\geq h'(\rho_N) + \phi_h'(\rho_N)\langle\rho^{-2}\nabla_N\rho_N, \alpha(e,e)\rangle_P \\ &\geq h(\rho_N)\left[\frac{h'(\rho_N)}{h(\rho_N)} - \rho^{-1}|\alpha(e,e)|_P\right]. \qquad (2.13) \end{aligned}$$

Substituting (2.11) and (2.13) in (2.10) we obtain

$$\frac{\psi(\rho_L)+1}{k\psi(\rho_L)} + \frac{|\alpha(e,e)|}{k} \geq h(\rho_N)\left[\frac{h'(\rho_N)}{h(\rho_N)} - \rho^{-1}|\alpha(e,e)|_P\right],$$

which implies that

$$\left[\frac{1}{h(\rho_N)k} + \frac{1}{\rho}\right]|\alpha(e,e)| \geq \frac{h'}{h}(\rho_N) - \frac{\psi(\rho_L)+1}{kh(\rho_N)\psi(\rho_L)}. \tag{2.14}$$

Since $-G \leq b$, by the Sturm Comparison Lemma [30] $|\alpha_{x_k}(e,e)| > 0$. Using the Otsuki's Lemma and the Gauss equation we find $X, Y \in V_{x_k}$, $|X| \geq |Y| \geq 1$ such that $|\alpha_{x_k}(X,X)| = |\alpha_{x_k}(Y,Y)|$ and

$$K_M(x_k)(X,Y) - K_N(\varphi(x_k))(X,Y) \geq |\alpha_{x_k}\left(\frac{X}{|X|}, \frac{X}{|X|}\right)|^2 > 0,$$

which means that $\sup K_M - \inf K_N > 0$ if $b \leq 0$. On the other hand, if $b < 0$, we let $k \to +\infty$ to get

$$\sup \rho^{-2} K_M - \inf \rho^{-2} K_N \geq \left[\lim_{k\to+\infty} \frac{h'}{h}(\rho_N(z_k))\right]^2 > |b|.$$

2.2 Slice Theorem in Robertson-Walker Spaces

One basic problem on the study of spacelike hypersurfaces in spacetimes is the uniqueness of spacelike hypersurfaces with constant mean curvature and constant higher order mean curvature in certain spacetimes, such as the conformally stationary spacetimes. One especial case of these spaces are the generalized Robertson-Walker spacetimes. Following Alias and Colares [4], a generalized Robertson-Walker spacetime is a Lorentzian warped product $-I \times_\rho M^n$ with Riemannian fiber M^n and warping function ρ. In particular, when the Riemannian factor M^n has constant sectional curvature then $-I \times_\rho M^n$ is classically called a Robertson-Walker spacetime. In this section we are interested in the study of uniqueness of complete spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes.

The works of Alias, Brasil Jr and Colares [3] and Alias and Colares [4] initiated the study of hypersurfaces with constant higher order mean curvature in conformally stationary spacetimes. Recently, Alias, Impera and Rigoli [5] were deeper into this study considering the case of compact and complete spacelike hypersurfaces. In order to obtain our second application we begin with the following tecnichal Lemma (see [5] for a proof).

Lemma 2.2. *Let $f\colon M^n \to -I \times_\rho N^n$ be a spacelike hypersurface immersed into a generalized Robertson-Walker spacetime. Let h be the height function and define $L_k = \mathrm{tr}\,(\mathrm{P}_k \circ \mathrm{hess}\,)$. Then*

$$L_k h = -\mathcal{H}(h)\,(c_k H_k + \langle \mathrm{P}_k \nabla h, \nabla h\rangle) - c_k \Theta H_{k+1}, \tag{2.15}$$

where $c_k = (n-k)\begin{pmatrix} n \\ k \end{pmatrix}$, $\Theta = \langle \eta, T\rangle$ and $\mathcal{H}(h) = \dfrac{\rho'(h)}{\rho(h)}$. In particular,

$$\Delta h = -\mathcal{H}(h)\left(n + |\nabla h|^2\right) - n\Theta H_1. \tag{2.16}$$

The next Lemma reveals us the important role of the behaviour of the angle function Θ in this theory.

Lemma 2.3. *Let $f\colon M^n \to -I \times_\rho N^n$ be an oriented spacelike hypersurface with $H > 0$, and assume that $\mathcal{H}' \le 0$. Suppose that M supports a Khas'minskii potential γ and that the height function h satisfies*

$$\lim_{x\to\infty} \frac{h(x)}{\varphi(\gamma(x))} = 0, \tag{2.17}$$

where φ *is given in* (1.4). *Then we have that*

i) If $\Theta < 0$ *then* $\mathcal{H}(h) \geq 0$*,*

ii) If $\Theta > 0$ *then* $\mathcal{H}(h) \leq 0$.

Proof. By hypothesis, we have that Omori-Yau maximum principle for the Laplacian holds for the height function h, therefore, there exist sequences $\{x_j\}, \{y_j\} \subset M$ such that

$$\lim_{j\to+\infty} h(x_j) = h^*, \quad |\nabla h|^2(x_j) < \left(\frac{1}{j}\right)^2, \quad \Delta h(x_j) < \frac{1}{j}, \tag{2.18}$$

and

$$\lim_{j\to+\infty} h(y_j) = h_*, \quad |\nabla h|^2(y_j) < \left(\frac{1}{j}\right)^2, \quad \Delta h(x_j) > -\frac{1}{j}. \tag{2.19}$$

Therefore, supposing $\Theta < 0$, we get by (2.16) and (2.18) that

$$\begin{aligned} \frac{1}{j} &> -\mathcal{H}(h(x_j))(n + |\nabla h|^2(x_j)) - nH_1(x_j)\Theta(x_j) \\ &\geq -\mathcal{H}(h(x_j))(n + |\nabla h|^2(x_j)). \end{aligned}$$

Then, when $j \to +\infty$ we obtain

$$0 \leq \limsup_{j\to+\infty} \mathcal{H}(h(x_j)) = \mathcal{H}(h^*) \leq \mathcal{H}(h). \tag{2.20}$$

Similar proof using (2.19) gives the item ii). $\square$

In the next theorem we present a slight generalization of the theorem 4.6 in [5]. In fact, now we do not make hypothesis about the growth of the height function, and nothing is required over the mean curvature H_1.

Theorem 2.2. *Let $f : M^n \to -I \times_\rho N^n$ be a complete oriented space-like hypersurface of constant 2-mean curvature $H_2 > 0$. Assume that*

$$K_M^{rad} \geq -F^2(r) \quad \textit{outside a compact set}, \tag{2.21}$$

where F is even at the origin and satisfies (1.1) *in Theorem 1.2. Suppose also that* h *satisfies the condition* (2.17). *If $\mathcal{H}' < 0$ almost everywhere, then $f(M)$ is a slice.*

Proof. Take an orientation on M in such a way that $H > 0$. If we assume that $\Theta < 0$, Lemma 2.3 implies that $\mathcal{H}(h) \geq 0$ and applying the Omori-Yau maximum principle for the Laplacian we have that there exists a sequence $\{x_j\} \subset M$ such that

i. $\lim_{j \to +\infty} h(x_j) = h^*$,

ii. $|\nabla h|(x_j) < \frac{1}{j}$,

iii. $\Delta h(x_j) < \frac{1}{j}$.

Since $\Theta(x_j) \leq -1$, the item iii. give us

$$\begin{aligned} \frac{1}{j} &> \; = -\mathcal{H}(h(x_j))\left(n + |\nabla h|^2(x_j)\right) - n\Theta(x_j) H_1(x_j) \\ &\geq \; -\mathcal{H}(h(x_j))\left(n + |\nabla h|^2(x_j)\right) + nH_2^{\frac{1}{2}} \end{aligned}$$

where we used the Garding's inequality in the last line. Making $j \to +\infty$, we get

$$\mathcal{H}(h^*) \geq H_2^{\frac{1}{2}}. \tag{2.22}$$

Analogously, applying the Omori-Yau maximum principle for $\hat{L}_1$ there exists a sequence $\{y_j\} \subset M$ such that

i. $\lim_{j\to+\infty} h(y_j) = h_*$,

ii. $|\nabla h|(y_j) < \frac{1}{j}$,

iii. $\hat{L}_1 h(y_j) > -\frac{1}{j}$.

Since P_1 is positive semi-definite, using the Garding's inequality again

$$\begin{aligned}
-\frac{1}{j} &< -\mathcal{H}(h(y_j))\left(n+\frac{1}{H_1}\langle P_1\nabla h,\nabla h\rangle(y_j)\right)\\
&\quad -n\Theta(y_j)\frac{H_2}{H_1}(y_j)\\
&\le -\mathcal{H}(h(y_j))\left(n+\frac{1}{H_1}\beta|\nabla h|^2(y_j)\right)-n\Theta(y_j)H_2^{\frac{1}{2}}\\
&\le -n\mathcal{H}(h(y_j))-n\Theta(y_j)H_2^{\frac{1}{2}}.
\end{aligned}$$

Therefore taking the limit in $j\to+\infty$

$$H_2^{\frac{1}{2}} \ge \mathcal{H}(h_*). \tag{2.23}$$

Combining (2.22) with (2.23), we get that

$$\mathcal{H}(h^*) \ge H_2^{\frac{1}{2}} \ge \mathcal{H}(h_*)$$

and as $\mathcal{H}$ is a decreasing function, we conclude that $h^* = h_* < \infty$. The case $\Theta > 0$ can be treated inverting the inequalities in the above proof. □

Using Theorem 1.3 we can extend Theorem 2.2 for the case of higher order mean curvatures.

Theorem 2.3. *Let $f: M^n \to -I\times_\rho N^n$ be an oriented complete space-like hypersurface of constant k-mean curvature $H_k > 0$, $3 \le k \le n$.*

Assume that

$$K_M^{rad} \geq -F^2(r) \quad \textit{outside a compact set}, \tag{2.24}$$

where F is even at the origin and satisfies (1.1) *in Theorem 1.2. Suppose also that the height function satisfies condition* (2.17) *and that there exists an elliptic point in M. If $\mathcal{H}' < 0$ almost everywhere, then $f(M)$ is a slice.*

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An alternating minimization method on geodesic spaces

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Dedicated to Professor Barnabé Pessoa Lima in honor of his 60^{th} birthday.

Abstract: In this note, we present an alternating minimization method on CAT(0) spaces for solving unconstrained convex optimization problems where the objective function is written as the sum of two separable convex functions. We prove that the sequence generated by our algorithm weakly converges to a minimizer of the objective function. The method proposed here is attractive to solve certain range of problems.

1 Introduction

Several convex optimization problems that arise in practice are modeled as the sum of convex functions, see for instance Goldfarb and Ma [17], Tikhonov and Arsenin [29], Candès et al. [10] and Donoho [16]. Minimizing sum of simple functions or finding a common point to a collection of closed sets is a very active field of research with application for instance in approximation theory (von Neumann [31]) and image reconstruction (Combettes

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and Wajs [13]). Let us describe some problems involving sum of convex functions.

- In *[18]*, Goldfarb et al. present the following problem:

$$\min_{x\in\mathbb{R}^n}\{F(x)=f(x)+g(x)\},$$

  where $f,g:\mathbb{R}^n\to\mathbb{R}$ are convex functions. A more specific example of this problem is the ℓ_1 minimization in compressed sensing (CS):

$$\min_{x\in\mathbb{R}^n}\left\{F(x)=\frac{1}{2}\|Ax-b\|_2^2+\rho\|x\|_1\right\},$$

  where $f(x)=\frac{1}{2}\|Ax-b\|_2^2$, $g(x)=\rho\|x\|_1$, $A\in\mathbb{R}^m\times\mathbb{R}^n$, $b\in\mathbb{R}^m$ and $\rho>0$.

- In *[13]*, the authors consider the following problem:

$$\min_{x\in\mathcal{H}}\{F(x)=f(x)+g(x)\},$$

  where $\mathcal{H}$ is a Hilbert space, $f:\mathcal{H}\to]-\infty,\infty]$ and $g:\mathcal{H}\to\mathbb{R}$ are proper, lower semicontinuous and convex functions, and g is differentiable on $\mathcal{H}$ with $1/\beta$-Lipschitz continuous gradient for some $\beta\in]0,\infty[$.

- In *[2]*, Attouch et al. describe a separable convex optimization problem with coupling as follows:

$$\min_{x\in\mathcal{H}_1,\ y\in\mathcal{H}_2}\{f(x)+g(y)+h(x,y)\},$$

  where $\mathcal{H}_1$,$\mathcal{H}_2$ are Hilbert spaces, $f:\mathcal{H}_1\to\mathbb{R}\cup\{+\infty\}$ and $g:\mathcal{H}_2\to\mathbb{R}\cup\{+\infty\}$ are proper, lower semicontinuous and convex functions, and $h:\mathcal{H}_1\times\mathcal{H}_2\to\mathbb{R}_+$ is a nonnegative quadratic form.

It is well known that the class of Proximal Point Algorithm (PPA) is one of the most studied methods for finding zeros of maximal monotone operators and, in particular, it is used to solve convex optimization problems. The classical PPA was introduced into optimization literature by Martinet [25]. It is based on the notion of proximal mapping J_λ^f,

$$J_\lambda^f(x) = \arg\min_z \{f(z) + \frac{1}{2\lambda}\|z-x\|^2\}, \tag{1.1}$$

introduced earlier by Moreau [26]; see also Brézis and Lions [9]. The PPA was popularized by Rockafellar [27], who showed the algorithm converges even if the auxiliary minimizations in (1.1) are performed inexactly, which is an important consideration in practice.

Gradually, many of the algorithms for solving optimization problems have been generalized from linear spaces (Euclidean, Hilbert, Banach) into differentiable manifolds. In particular, the proximal point algorithm in the context of Riemannian manifolds (of nonpositive sectional curvature) was studied for instance in [7, 19, 24, 28]. Along these lines Băcák [5] introduced the PPA into geodesic metric spaces of nonpositive curvature, so-called CAT(0) spaces. Recently, Zaslavski [33] proposed a different approach to the PPA in metric spaces; see also [1, 11, 12]. The main advantages of these extensions are that nonconvex problems in the classic sense may become convex and constrained optimization problems may be seen as unconstrained ones through the introduction of an appropriate Riemannian metric; see [15, 20, 30].

As mentioned by Băcák [5] there are natural obstacles one has to overcome in CAT(0) spaces. Unlike Riemannian manifolds, CAT(0) spaces do not come equipped with a Riemannian metric, and, probably relatedly, we do not have a notion of a subgradient of a convex function. The proof of (weak) convergence of the PPA in Hilbert spaces, on the other hand, does use both the inner product and the convex subgradient, and therefore we cannot simply translate the existing proof into the context of metric spaces.

In this note, we present an alternating minimization method for solving unconstrained convex optimization problems where the objective function is written as the sum of two separable convex functions. Alternating algorithms has been studied in several settings. The starting fundamental result is due to von Neumann [32] with the purpose of solving convex feasibility problems. More recently Attouch et al. [3] presented alternating algorithms for nonconvex functions with applications in decision sciences. Other important approach of alternating algorithm with applications in game theory in the context of Hadamard manifolds was presented by Cruz Neto et al. [14]. One of the main advantage of alternating algorithms is that it enables us to monitor what happens in each space of action after a given iteration. Furthermore, computations are quite simplified compared to the one in the product space. Due to the wide potential range of applications of alternating algorithms (from engineering to decision sciences) we adopt a quite general terminology.

2 Elements of CAT(0) spaces

In this section we introduce some fundamental properties and notations concerning CAT(0) spaces which can be found for instance in Bridson and Haefliger [8].

Let (M,d) be a metric space, where M is a set and d a metric in M. A geodesic path joining $x \in M$ to $y \in M$ (or, more briefly, a geodesic from x to y) is a map γ from a closed interval $[0,\ell] \subset \mathbb{R}$ such that $\gamma(0) = x$, $\gamma(\ell) = y$ and $d(\gamma(t),\gamma(t')) = |t-t'|$ for all $t,t' \in [0,\ell]$. Let M be a geodesic space, i.e., a metric space for which every two points $x,y \in M$ can be joined by a geodesic segment, and $\Delta(x,y,z)$ a geodesic triangle in M, which is a union of three geodesics. Let $[x,y]$ denote the geodesic side between x,y. A comparison triangle for Δ is a triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ in $\mathbb{R}^2$ with the same side lengths as Δ. The interior angle of $\bar{\Delta}$ at $\bar{x}$ is called the comparison angle between y and z at x, and is denoted $\alpha'(y,x,z)$. Let p be a point on a side of Δ, say, $[x,y]$. A comparison point in $\bar{\Delta}$ is a point $\bar{p} \in [\bar{x},\bar{y}]$ with $d(x,p) = d_{\mathbb{R}^2}(\bar{x},\bar{p})$. $\bar{\Delta}$ satisfies the $CAT(0)$ inequality if for any $p,q \in \Delta$ and their comparison points $\bar{p},\bar{q} \in \bar{\Delta}$,

$$d(p,q) \leq d_{\mathbb{R}^2}(\bar{p},\bar{q}).$$

M is a CAT(0) space iff all geodesic triangles in M satisfy the CAT(0) inequality. (M,d) is said to be a geodesic metric space (or, more briefly, a geodesic space) if every two points in M are joined by a geodesic.

Let (M,d) be a $CAT(0)$ space. Having two points $x,y \in M$, we denote the geodesic segment from x to y by $[x,y]$. We usually do not distinguish between a geodesic and its geodesic segment, as no confusion can arise. A subset A of a metric space (M,d) is said to be convex iff every pair of points $x,y \in A$ can be joined by a geodesic in M and the image of every such geodesic is contained in A.

Example 2.1. *Standard examples of* CAT(0) *spaces, see* [8].

- *Hyperbolic spaces,* $\mathbb{H}^n$..
- *Symmetric spaces of non-compact type. For example,* $SL(n,\mathbb{R})/SO(n)$.
- *Hadamard manifolds, i.e., complete, simply connected Riemannian manifolds of non-positive sectional curvature.*
- *Products of* CAT(0) *spaces.*
- *When endowed with the induced metric, a convex subset of Euclidean space* $\mathbb{R}^n$ *is* CAT(0).
- *Euclidean space,* $\mathbb{R}^n$.

Remark 11. *From [5] a geodesic metric space* (M,d) *is a* CAT(0) *space if for any* $x \in M$ *and* $t \in [0,1]$, *and any geodesic* $\gamma : [0,1] \to M$, *we have*

$$d^2(x,\gamma(t)) \leq (1-t)d^2(x,\gamma(0)) + td^2(x,\gamma(1)) - t(1-t)d^2(\gamma(0),\gamma(1)). \tag{2.1}$$

For any metric space (M,d) and $A \subset M$, we define the distance function by

$$d_A(x) = \inf_{a \in A} d(x,a), \; x \in M.$$

Let us note that the function d_A is convex and continuous provided M is CAT(0) space and A is a convex and complete set, see [8, page 178].

Lemma 2.1. *Let* (M,d) *be a* CAT(0) *space and* $A \subset M$ *be complete and convex. Then,*

(i) *For every* $x \in M$, *there exists an unique point* $P_A(x) \in A$ *such that* $d(x,P_A(x)) = d_A(x)$.

(ii) *If* $y \in [x,P_A(x)]$, *then* $P_A(x) = P_A(y)$.

(iii) If $x \in M \setminus A$ *and* $y \in A$ *such that* $P_A(x) \neq y$*, then*

$$\alpha(x, P_A(x), y) \geq \pi/2.$$

(iv) The mapping P_A *is a non-expansive retraction from* M *onto* A*.*

Proof. See [8, page 176]. □

The mapping P_A is called the metric projection onto A. A point $x^0 \in M$ is called the weak limit of a sequence $\{x^n\}_{n \in \mathbb{N}} \subset M$ iff for every geodesic arc γ starting at x^0, $P_\gamma(x^n)$ converges to x^0. In this case, we say that $\{x^n\}$ converges to x^0 weakly. We use the notation $x^n \xrightarrow{w} x^0$. Note that if $x^n \to x^0$, then $x^n \xrightarrow{w} x^0$. Moreover, if there is a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that $x^{n_k} \xrightarrow{w} \bar{x}$ for some $\bar{x} \in M$, we say that $\bar{x}$ is a weak cluster point of the sequence $\{x^n\}$. Every bounded sequence has a weak cluster point, see [21, Theorem 2.1] or [22, page 3690]. A function $f : M \to (-\infty, \infty]$ is called weakly lower semicontinuous at a given point $x \in M$ iff

$$\liminf_{n \to \infty} f(x^n) \geq f(x)$$

for each sequence $x^n \xrightarrow{w} x$. A sequence $\{x^n\} \subset M$ is Fejér monotone with respect to $A \subset M$ if, for any $a \in A$,

$$d(x^{n+1}, a) \leq d(x^n, a), \quad n \in \mathbb{N}.$$

3 Alternating proximal algorithm

In this note, we propose and analyze an alternating proximal algorithm in the setting of CAT(0) spaces. Consider the following minimization problem

$$\begin{aligned} &\min H(x,y) \\ &s.t.\ (x,y) \in M \times N, \end{aligned} \tag{3.1}$$

where M and N are CAT(0) spaces and $H : M \times N \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function bounded from below which has the following structure:

(i) $H(x,y) = f(x) + g(y)$;

(ii) $f : M \to \mathbb{R} \cup \{+\infty\}$, $g : N \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semicontinuous.

The alternating proximal algorithm to solve optimization problems of the form (3.1) generates, for a starting point $z^0 = (x^0, y^0) \in M \times N$, a sequence $\{z^k\}_{k\in\mathbb{N}}$, with $z^k = (x^k, y^k) \in M \times N$ updated as follows:

$$(x^k, y^k) \rightsquigarrow (x^{k+1}, y^k) \rightsquigarrow (x^{k+1}, y^{k+1})$$

$$\begin{cases} x^{k+1} = \arg\min\{H(x, y^k) + \lambda_k d_M^2(x^k, x);\ x \in M\} \\ \\ y^{k+1} = \arg\min\{H(x^{k+1}, y) + \mu_k d_N^2(y^k, y);\ y \in N\}, \end{cases} \tag{3.2}$$

where d_M, d_N are distances associated with spaces M and N respectively, $\{\lambda_k\}$ and $\{\mu_k\}$ are sequences of positive numbers and $H(x,y) = f(x) + g(y)$ is a separable function. Previous related works can be found in Attouch et al. [2, 3, 4], in case that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. In [23], Lewis and Malick studied a method of alternating projections in which $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^n$ are two smooth manifolds transversally intersect.

At each iteration the method solves the following subproblems in which we apply the proximal point method to solve them:

$$\min_{x\in M}\left\{H(x,y^k)+\lambda_k d_M^2(x^k,x)\right\} \tag{3.3}$$

and

$$\min_{y\in N}\left\{H(x^{k+1},y)+\mu_k d_N^2(y^k,y)\right\}. \tag{3.4}$$

4 Convergence analysis

We start with an useful result whose the proof follows easily from the fact that a closed convex subset of a complete CAT(0) space is (sequentially) weakly closed [6, Lemma 3.1].

Proposition 4.1. *Let M be a complete* CAT(0) *space. If $f : M \to (-\infty,\infty]$ is a lower semicontinuous and convex function, then it is weakly lower semicontinuous.*

Proof. See [5, Lemma 3.1]. □

We define the distance d in $M\times N$ as follows:

$$d(z_1,z_2)=\left(d_M^2(x_1,x_2)+d_N^2(y_1,y_2)\right)^{1/2},$$

for all $z_1=(x_1,y_1),\ z_2=(x_2,y_2)$ in $M\times N$, where d_M and d_N are distances in M and N respectively. It is easy to check that d is a distance in $M\times N$.

Now, let $A\subset M\times N$ be the set of minimizers of H, i.e.,

$$A:=\left\{a\in M\times N;\ H(a)=\inf_{a\in M\times N}H(z)\right\}.$$

Without loss of generality we can assume that $H(a)=0$, for all $a\in A$.

Lemma 4.1. *Let* $\{z^k\}$ *be the sequence generated by* (3.2) *and* $a \in A$. *Then,*

(i) $d(z^{k+1}, a) \le d(z^k, a), \ \forall k \in \mathbb{N}$;

(ii) $(\bar{c}_k)^{-1} H(z^{k+1}) \le d^2(z^k, a) - d^2(z^{k+1}, a), \ \bar{c}_k > 0$.

Proof. Let $\gamma : [0,1] \to M \times N$ be a geodesic, with $\gamma(0) = a$ and $\gamma(1) = z^{k+1}$. From (3.2),

$$f(x^{k+1}) + g(y^k) + \lambda_k d_M^2(x^k, x^{k+1}) \le f(x) + g(y^k) + \lambda_k d_M^2(x^k, x)$$

and

$$f(x^{k+1}) + g(y^{k+1}) + \mu_k d_N^2(y^k, y^{k+1}) \le f(x^{k+1}) + g(y) + \mu_k d_N^2(y^k, y).$$

Adding last two inequalities, we obtain

$$H(z^{k+1}) + c_k d^2(z^k, z^{k+1}) \le H(z) + c_k' d^2(z^k, z), \tag{4.1}$$

where $c_k = \min\{\lambda_k, \mu_k\}$ and $c_k' = \min\{\lambda_k, \mu_k\}$. Thus,

$$H(z^{k+1}) + c_k d^2(z^k, z^{k+1}) \le H(\gamma(t)) + c_k' d^2(z^k, \gamma(t)).$$

The convexity of H means that

$$H(\gamma(t)) \le (1-t) H(\gamma(0)) + t H(\gamma(1)),$$

which implies

$$c_k d^2(z^k, z^{k+1}) - c_k' d^2(z^k, \gamma(t)) \le H(\gamma(t)) - H(z^{k+1}) \le (t-1) H(z^{k+1}).$$

Taking $x = z^k$ in inequality (2.1), we get

$$t(1-t)d^2(z^{k+1},a)-(1-t)d^2(z^k,a) \le td^2(z^k,z^{k+1})-d^2(z^k,\gamma(t)).$$

Combining last two inequalities, we have

$$t(1-t)d^2(z^{k+1},a)-(1-t)d^2(z^k,a) \le \frac{1}{\bar{c}_k}(t-1)H(z^{k+1}) \le 0, \qquad (4.2)$$

where $\bar{c}_k = \max\{c_k, c_k'\}$ and $t \neq 1$. Therefore,

$$td^2(z^{k+1},a)-d^2(z^k,a) \le 0.$$

Taking $t=1$ in last inequality we prove item (i). Item (ii) is a direct consequence of (4.2). □

Theorem 4.1. *Let* $(M\times N, d)$ *be a complete* CAT(0) *space and* H *be a lower semicontinuous and convex function. Assume that* H *has a minimizer. Then, for a starting point* $z^0 \in M\times N$, *and a sequence of positive numbers* $\{\bar{c}_k\}$ *such that* $\sum_1^\infty (\bar{c}_k)^{-1} = \infty$, *the sequence* $\{z^k\} \subset M\times N$ *defined by* (3.2) *weakly converges to a minimizer of* H.

Proof. From Lemma 4.1 (i) and (ii), we have

$$H(z^{j+1})\sum_{k=1}^{j-1}(\bar{c}_k)^{-1} \le \sum_{k=1}^{j-1}(\bar{c}_k)^{-1}H(z^{k+1}) \le \frac{1}{2}d^2(z^1,a)-\frac{1}{2}d^2(z^j,a)$$

and

$$H(z^{j+1}) \le \frac{d^2(z^1,a)}{\sum_{k=1}^{j-1}(\bar{c}_k)^{-1}}.$$

Since the right-hand side of last inequality goes to zero as $j\to\infty$, it follows that $\{z^k\}$ is a minimizing sequence. Therefore, $\{H(z^k)\}$ converges to zero as $k\to\infty$. To finish the proof, let z^* be a weak cluster point of $\{z^k\}$. From Proposition 4.1, H is weakly lower semicontinuous. Thus, $H(z^*) = 0$, and therefore $z^* \in A$. Using [6, Proposition 3.3] it follows $z^k \xrightarrow{w} z^*$ and the proof is completed.

$\square$

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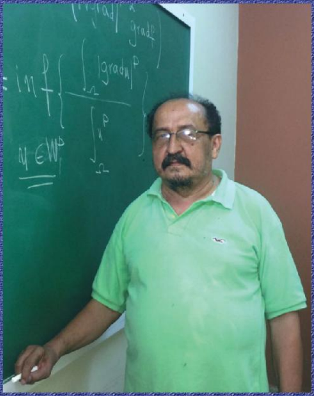

Born in Beneditinos, Piauí, on the banks of the Poty River, he lived until he was 12 years old. His thirst for knowledge made him follow the flow of the Poty to the capital Teresina, where he has lived ever since. He was licensed in General Sciences with a focus in Mathematics from Universidade Federal do Piauí in 1984. He obtained his Master's degree in Mathematics (1990) from IMPA, and his Doctorate degree (2000) from UFC. He has been a Professor at UFPI since 1985, where he divides his time between classes, advising and research in geometric analysis, always accompanied, of course, by a good coffee. Great lover of good beer, he has fun playing billiards, and watching (suffering through) Fluminense soccer games.

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