# NONLOCAL FINITE ELEMENT ANALYSIS FOR FREE VIBRATION OF ELASTICALLY SUPPORTED NANOBEAMS

**Aldemar P. Siqueira Neto**
**Simone S. Hoefel**
*aldemarpsiqueiran@ufpi.edu.br*
*simone.santos@ufpi.edu.br*
*Computational Modeling Methods Laboratory, Federal University of Piauí*
*Campus Universitário Ministro Petrônio Portella - Ininga, 64049-550, Teresina PI, Brazil*

**Abstract.** Nanobeams are nanoscale structures extensively used in nanotechnology applications. Due to the small scale effect, this nanostructures cannot be accurately modelled by traditional elastic theory. To overcome this difficulty, several continuous models including the material length scale effect were developed, like nonlocal elasticity theory. In this paper, a nonlocal finite element model for elastically supported Euler-Bernoulli (EBT) and Timoshenko (TBT) nanobeams is developed. Nonlocal differential constitutive equations of Eringen are considered to account for the small scale effect. The stiffness and mass matrices for a two-node nonlocal beam element with two degrees of freedom per node are obtained based upon Hamilton's principle. The influence of nonlocal parameter, slenderness ratio and support stiffness on the free vibration characteristics is investigated. Numerical results obtained are discussed and compared with results obtained by other researchers.

**Keywords:** Elastically supported nanobeams, Nonlocal elasticity, Nonlocal finite element method, Free vibration.

## 1 Introduction

Nanotechnology has been greatly advanced over the last decades enabling the use of nano-sized structures in many engineering and industrial applications such as nanowires, nanobridges, nanotubes and nanosensors. Understanding the mechanical behaviour of this structures is extremely important for efficient design and fabrication of nanoelectromechanical devices. Theoretical analysis is widely used for many researchers in this task, given the difficulty to control experiments in nano scale. Continuum models are generally used in contrast to atomistic models which are restricted to small length and time scales due to being computationally expensive. However, classical elasticity theory is scale independent and cannot be directly applied to model this extremely small structures. This fact led to the development of various modified continuum models to account for the small scale effects such as the couple stress theory, strain gradient elasticity theory, Cosseract elasticity and nonlocal elasticity theory.

The most widely used non-classical theory is nonlocal elasticity theory developed by Eringen and Edelen [1], Eringen [2]. It assumes that the stress at a reference point in a continuum body is a function of the strains at all points of the body. This definition introduces a nonlocal parameter into the constitutive equations which accounts for the lattice dynamics and internal length scale effect on the material elastic behaviour. A large amount of research activities based on nonlocal elasticity theory have taken place over the last years. In 2006, the influence of nonlocal effect on transverse free vibration of nano-to-micro scale beams was investigated by Xu [3]. He found that for beams at micrometre scale length, the classical Euler-Bernoulli theory is applicable, in contrast to beams at nanometre scale, where the nonlocal effects become significant, especially for higher-order frequencies and vibrating modes. Reddy [4] reformulated local beam theory by using the nonlocal differential constitutive relations of Eringen and utilized Fourier series to obtain analytical solutions for bending, buckling and vibration of simply supported nanobeams. Civalek et al. [5] utilized the differential quadrature method (DQM) to investigate the small-scale effect on bending and free vibration of microtubules. Vibration analysis of multi-walled carbon nanotubes (MWCNTs) was studied by Ehteshami and Hajabasi [6] utilizing a multiple nonlocal Euler-Bernoulli beam model.

In 2012, Pradhan [7] reported finite element formulations for nonlocal EBT and TBT and presented results for bending, buckling and vibration. Aydogdu [8] reported a explicit equation for the nonlocal coefficient based in the longitudinal wave propagation in nanorods. Eltaher et al. [9] used the finite element method (FEM) to investigate the effect of nonlocal parameter on the fundamental frequencies of nanobeams for various classical boundary conditions. Nazemnezhad and Hosseini-Hashemi [10] presented analytical solutions for nonlocal nonlinear free vibrations of functionally graded nanobeams using Euler-Bernoulli theory with von kármán type nonlinearity. Dynamic characteristics of single walled carbon nanotubes (SWCNT) was investigated by Boumia et al. [11] utilizing the Timoshenko beam model and nonlocal continuum theory. They observed that chirality of carbon nanotubes and small scale effect have significant influence on the free vibration frequencies. Rahmanian et al. [12] applied nonlocal elasticity theory to study the free vibration characteristics of SWCNTs resting on a Wrinkler foundation with EBT and Love shell models. Results obtained by series expansion showed that shell model presents higher accuracy than beam model for nanoscale analysis and that natural frequencies converge to the clamped boundary conditions as the foundation stiffness parameters increases. More recently, Demir and Civalek [13] developed a Galerkin finite element model to study thermal vibration of nanobeams surrounded by an elastic matrix. The numerical results show that the thermal effect decreases the free vibration frequencies while the inclusion of elastic matrix effect increases the frequencies.

The effect of support stiffness on the dynamic behaviour of local beams have been extensively studied for various researchers. The free vibration of an Euler-Bernoulli beam with one free end and the other hinged with a rotational spring was studied by Chun [14] in 1972. Abbas [15] employed finite element model which can satisfy all geometric and natural boundary conditions to solve the problem of free vibration of elastically supported Timoshenko beams. Craver Jr. and Jampala [16] investigated the free vibration of an Euler-Bernoulli linearly tapered beam elastically constrained. In 2005, analytical solutions for free vibrations of elastically supported Timoshenko beams was presented by Kocatürk and

Şimşek [17] using Lagrange equations with trial functions in power series form. Azevêdo et al. [18] presented analytical and numerical results (FEM) for elastically supported beams based on EBT and TBT. Concerning with elastically supported nonlocal structures, a recent work by Kiani [19] presented a novel integro-differential surface energy-based model and studied the influence of support stiffness, nonlocality, surface energy and kernel function on longitudinal free vibration characteristics of elastically supported nanorods.

In this paper, finite element formulation for elastically supported Euler-Bernoulli and Timoshenko nanobeams is developed using nonlocal constitutive differential relations. Differential motion equations are presented. The stiffness and mass matrices for a two-node nonlocal beam element with two degree of freedom per node is obtained based upon Hamilton's principle. The influence of nonlocal parameter, slenderness ratio and support stiffness on the free vibration characteristics is investigated. For this purpose, the first three non-dimensional natural frequencies are calculated for various rigidity values of translational and rotational springs. Numerical results obtained are discussed and compared with results obtained by other researchers.

## 2 Review of beam theories

The description of the beam theories was developed considering the following coordinate system: $x$-axis, $z$-axis and $y$-axis are taken along the length, height and width of the beam, respectively. All applied loads and geometry are assumed such that the displacements ($u_1$, $u_2$, $u_3$) along the coordinates ($x$, $y$, $z$) are functions only of $x$ and $z$ coordinates and time $t$. Displacement $u_2$ is considered identically zero.

### 2.1 Euler-Bernoulli beam theory

The Euler-Bernoulli beam theory (EBT) is based in the following displacement field:

$$u_1 = u(x,t) - z\frac{\partial w^E}{\partial x}, \quad u_2 = 0, \quad u_3 = w^E(x,t), \tag{1}$$

where $u$ and $w^E$ are the axial and transverse displacements at a point on the mid-plane of the beam (z = 0). The superscript 'E' denotes the quantities in Euler-Bernoulli beam. The only nonzero strain of EBT is

$$\varepsilon_{xx}^{\mathrm{E}} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w^{\mathrm{E}}}{\partial x^2} \equiv \varepsilon_{xx}^0 + z\kappa^{\mathrm{E}}, \quad \varepsilon_{xx}^0 = \frac{\partial u}{\partial x}, \quad \kappa^{\mathrm{E}} = -\frac{\partial^2 w^{\mathrm{E}}}{\partial x^2} \tag{2}$$

where $\varepsilon_{xx}$ is the normal strain, $\varepsilon_{xx}^0$ is the extensional strain and $\kappa^{\mathrm{E}}$ is the bending strain. The stress resultant $M$ is defined as

$$M = \int_A z\sigma_{xx} dA, \tag{3}$$

in which $\sigma_{xx}$ denotes the normal stress. Neglecting the normal strain contribution, the Hamilton's principle for transverse free vibrations of an Euler-Bernoulli beam has the following form (Reddy [4]):

$$0 = \int_0^{\mathrm{T}} \int_0^{\mathrm{L}} \left[ M^{\mathrm{E}} \delta\kappa^{\mathrm{E}} - I_0 \frac{\partial w^{\mathrm{E}}}{\partial t} \delta \frac{\partial w^{\mathrm{E}}}{\partial t} \right] \mathrm{d}x\mathrm{d}t, \tag{4}$$

where T is an arbitrary instant of time, L is the length of the beam, $\delta(\,)$ denotes a virtual change and the mass inertia is given by

$$I_0 = \int_A \rho dA, \tag{5}$$

in which $\rho$ is the mass per unit volume. The following Euler-Lagrange equation in terms of stress resultant is obtained for $0 < x < \mathrm{L}$

$$\frac{\partial^2 M^E}{\partial x^2} = I_0 \frac{\partial^2 w^E}{\partial t^2}\,, \tag{6}$$

### 2.2 Timoshenko beam theory

Timoshenko beam theory (Timoshenko [20]) includes both effects of shear deformation and rotatory inertia to Euler-Bernoulli beam and is based on the displacement field

$$u_1 = u(x,t) - z\phi^{\mathrm{T}}, \quad u_2 = 0, \quad u_3 = w^T(x,t), \tag{7}$$

in which $\phi^{\mathrm{T}}$ is the cross-section rotation and the superscript 'T' refers to Timoshenko beam quantities. The nonzero strains of TBT are

$$\begin{aligned} \varepsilon^{\mathrm{T}}_{xx} &= \frac{\partial u}{\partial x} - z\frac{\partial \phi^{\mathrm{T}}}{\partial x} \equiv \varepsilon^0_{xx} + z\kappa^{\mathrm{T}}, \quad 2\varepsilon^{\mathrm{T}}_{xz} = \frac{\partial w^{\mathrm{T}}}{\partial x} - \phi^{\mathrm{T}} \equiv \gamma^{\mathrm{T}}, \\ \varepsilon^0_{xx} &= \frac{\partial u}{\partial x}, \quad \kappa^{\mathrm{T}} = -\frac{\partial \phi^{\mathrm{T}}}{\partial x}, \quad \gamma^{\mathrm{T}} = \frac{\partial w^{\mathrm{T}}}{\partial x} - \phi^{\mathrm{T}}, \end{aligned} \tag{8}$$

where $\kappa^{\mathrm{T}}$ and $\gamma^{\mathrm{T}}$ denotes the bending and transverse shear strains, respectively. The stress resultant $Q$ is defined as

$$Q = \int_A \sigma_{xz} dA, \tag{9}$$

where $\sigma_{xy}$ denotes the transverse shear stress. Neglecting the normal strain contribution, the Hamilton's principle for transverse free vibrations of a Timoshenko beam takes the following form (Reddy [4]):

$$0 = \int_0^{\mathrm{T}} \int_0^{\mathrm{L}} \left[ M^T \delta\kappa^T + Q^T \delta\gamma^T - I_0 \frac{\partial w^T}{\partial t} \delta \frac{\partial w^T}{\partial t} - I_2 \frac{\partial \phi^T}{\partial t} \delta \frac{\partial \phi^T}{\partial t} \right] \mathrm{d}x\mathrm{d}t, \tag{10}$$

where the rotatory inertia $I_2$ is given by

$$I_2 = \int_A \rho z^2 dA. \tag{11}$$

The Euler-Lagrange equations obtained in terms of stress resultants are

$$\frac{\partial Q^T}{\partial x} = I_0 \frac{\partial^2 w^T}{\partial t^2}, \tag{12}$$

$$Q^T - \frac{\partial M^T}{\partial x} = I_2 \frac{\partial^2 \phi^T}{\partial t^2}, \tag{13}$$

## 3 Nonlocal elasticity theory

Nonlocal elasticity theory developed by Eringen [2] assumes that the stress field on a reference point in a elastic continuum depends on the strain of all points in the body, differently of the hyperelastic case, where it only depends on the strain at the point of reference. Neglecting body forces, the nonlocal stress tensor $\sigma$ for a linear homogeneous nonlocal elastic body is written as (Xu [3])

$$\sigma = \int_V K(|x' - x|, \tau) t(x') dx', \tag{14}$$

where $t(x)$ is the macroscopic stress tensor at point $x$ and the kernel function $K(|x' - x|, \tau)$ represents the nonlocal modulus $|x' - x|$, being the distance (in Euclidean norm) and $\tau$ is a material constant that the depends on internal and external characteristics lengths. The constitutive equation relating the macroscopic stress at point $x$ to the strain $\varepsilon$ at the point is given by the generalized Hook's law:

$$t(x) = C(x) : \varepsilon(x), \tag{15}$$

in which C is the fourth-order elasticity tensor and : denotes the "double-dot product".

Alternatively to the integral constitutive relation in Eq. (14), whose solution is difficult, the following equivalent differential form, reported by Eringen [2], is used as basis of all nonlocal constitutive formulations:

$$(1 - \tau^2 {l_e}^2 \nabla^2)\sigma = t, \tag{16}$$

where $\tau = e_0 l_i / l_e$, $e_0$ is a material constant, $l_i$ and $l_e$, the internal and external characteristics lengths, respectively, and $\nabla^2$ is the Laplacian operator.

### 3.1 Stress resultants for nonlocal beams

Considering the one-dimensional geometric characteristic of beams, it is assumed that the effect of nonlocal behaviour is negligible in the thickness direction in comparison to longitudinal direction. Therefore, the Laplacian operator in Eq. 16 is reduced to one dimensional form and the nonlocal constitutive relation assumes the following forms for isotropic homogeneous beams:

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \quad \text{and} \tag{19.a}$$

$$\sigma_{xy} - \mu \frac{\partial^2 \sigma_{xy}}{\partial x^2} = 2G\varepsilon_{xy}, \tag{19.b}$$

where the nonlocal parameter is defined as $\mu = e_0^2 l_i^2$, $E$ and $G$ are the Young's modulus and shear modulus, respectively. For a nonlocal parameter $\mu$ equal to zero these equations reduce to the local constitutive relations.

The relations given by Eq. (19.a) and Eq. (19.b) yields the following stress resultant differential equations for Euler-Bernoulli and Timoshenko nonlocal beams:

$$M^E - \mu \frac{\partial^2 M^E}{\partial x^2} = EI\kappa^E, \tag{20}$$

$$M^T - \mu \frac{\partial^2 M^T}{\partial x^2} = EI\kappa^T, \quad Q^T - \mu \frac{\partial^2 Q^T}{\partial x^2} = K_s GA\gamma^T, \tag{21}$$

in which $I$ is the second moment of area about the $y$-axis and $K_s$ denotes the shear correction factor.

## 4 Free vibration governing equations in terms of displacements

The governing equations given by Eq. (6), Eq. (12) and Eq. (13) can be rewritten in terms of displacements using the relationships given by Eq. (20) and Eq. (21). For Euler-Bernoulli beam the nonlocal stress resultant $M^E$ assumes the form

$$M^E = -EI \frac{\partial^2 w^E}{\partial x^2} + \mu \left( I_0 \frac{\partial^2 w^E}{\partial t^2} \right), \tag{22}$$

and the transverse free vibration governing equation in terms of displacements is given by

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w^E}{\partial x^2} \right) + I_0 \frac{\partial^2 w^E}{\partial t^2} - \mu \frac{\partial^2}{\partial x^2} \left( I_0 \frac{\partial^2 w^E}{\partial t^2} \right) = 0. \tag{23}$$

For Timoshenko beam the nonlocal stress resultants $Q^T$ and $M^T$ are given by

$$Q^T = K_s GA \left( \frac{\partial w^T}{\partial x} - \phi^T \right) + \mu I_0 \frac{\partial^3 w^T}{\partial x \partial t^2}, \tag{24}$$

$$M^T = -EI \frac{\partial \phi^T}{\partial x} + \mu \left[ I_0 \frac{\partial^2 w^T}{\partial t^2} - I_2 \frac{\partial^3 \phi^T}{\partial x \partial t^2} \right], \tag{25}$$

and the transverse free vibration governing equations in terms of displacements are given by

$$\frac{\partial}{\partial x} \left[ K_s GA \frac{\partial w^T}{\partial x} - \phi^T \right) \right] - I_0 \frac{\partial^2 w^T}{\partial t^2} + \mu I_0 \frac{\partial^4 w^T}{\partial x^2 \partial t^2} = 0, \tag{26}$$

$$\frac{\partial}{\partial x} \left( EI \frac{\partial \phi^T}{\partial x} \right) + K_s GA \left( \frac{\partial w^T}{\partial x} - \phi^T \right) - I_2 \frac{\partial^2 \phi^T}{\partial t^2} + \mu I_2 \frac{\partial^4 \phi^T}{\partial x^2 \partial t^2} = 0. \tag{27}$$

Table 1 presents some boundary conditions for both theories that must be specified at both ends of the beam, where $k_m$ and $k_r$ are, respectively, the translational and rotational spring constant, $S_o = -1$ for $x = 0$ and $S_o = 1$ for $x =$ L.

Table 1. Boundary conditions

| Boundary Condition | $\frac{\partial M^E}{\partial x}$, $Q^T$ | $M^E$, $M^T$ | $\frac{\partial w^E}{\partial x}$, $\phi^T$ | $w^E$, $w^T$ |
|---|---|---|---|---|
| Hinged | - | 0 | - | 0 |
| Clamped | - | - | 0 | 0 |
| Free | 0 | 0 | - | - |
| Sliding | 0 | - | 0 | - |
| Linear Spring | $S_o k_m \cdot w^E$, $S_o k_m \cdot w^T$ | 0 | - | - |
| Torsional Spring | 0 | $S_o k_r \cdot \frac{\partial w^E}{\partial x}$, $S_o k_r \cdot \phi^T$ | - | - |

## 5 Finite element formulation

In order to develop the finite element model, the variational statements given by Eq. (4) and Eq. (10) can be rewritten in terms of the stress resultants in Eq. (20) and Eq. (21). The Hamilton's principle takes the following form for a Timoshenko nonlocal beam

$$\begin{aligned} 0 = \int_0^{\mathrm{T}} \int_0^{\mathrm{L}} \Bigg\{ & EI \frac{\partial \phi^T}{\partial x} \delta \frac{\partial \phi^T}{\partial x} + K_s GA \left( \frac{\partial w^T}{\partial x} - \phi^T \right) \delta \left( \frac{\partial w^T}{\partial x} - \phi^T \right) - I_0 \frac{\partial w^T}{\partial t} \delta \frac{\partial w^T}{\partial t} \\ & - I_2 \frac{\partial \phi}{\partial t} \delta \frac{\partial \phi}{\partial t} + \mu I_0 \left[ \frac{\partial^3 w^T}{\partial x \partial t^2} \delta \left( \frac{\partial w^T}{\partial x} - \phi^T \right) - \frac{\partial^2 w}{\partial t^2} \delta \frac{\partial \phi^T}{\partial x} \right] + \mu I_2 \frac{\partial^3 \phi}{\partial x \partial t^2} \delta \frac{\partial \phi^T}{\partial x} \Bigg\} \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{28}$$

The equivalent weak form for Euler-Bernoulli nonlocal beam can be obtained by substituting $\phi^T = \partial w^E / \partial x$ in Eq. (28) and neglecting the shear stress and rotatory inertia terms.

Consider a uniform Timoshenko beam element as shown in Fig. 1, with two nodes and two degrees of freedom per node: $W$, the total deflection, and $\Phi$, the bending slope. Using the non-dimensional coordinate, $\xi$, and element length $l_e$ defined in Fig. 1, the displacement and total slope can be written in matrix form as follows:

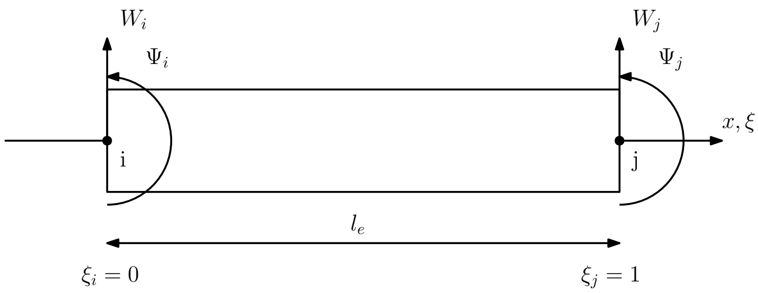

Figure 1. Nonlocal beam element

$$W = [\,\mathbf{N}(\xi)\,]\{\,\mathbf{v}\,\}_e \quad \text{and} \quad \Phi = [\,\overline{\mathbf{N}}(\xi)\,]\{\,\mathbf{v}\,\}_e, \tag{29}$$

where

$$[\mathbf{N}(\xi)] = [N_1(\xi) \quad N_2(\xi) \quad N_3(\xi) \quad N_4(\xi)] \quad \text{and} \tag{30}$$

$$[\overline{\mathbf{N}}(\xi)] = [\overline{N}_1(\xi) \quad \overline{N}_2(\xi) \quad \overline{N}_3(\xi) \quad \overline{N}_4(\xi)] \tag{31}$$

are the shape functions and $\{\, \mathbf{v}\, \}_e$ is the vector of nodal coordinates. The subscript $e$ represents expressions for a single element. The shape functions in Eq. (30) and Eq. (31) are written as

$$\mathbf{N}_\mathrm{i}(\xi) = \left(\frac{1}{\Gamma+1}\right) \begin{bmatrix} 2\xi^3 - 3\xi^2 - \Gamma\xi + \Gamma + 1 \\ (l_e/2)[2\xi^3 - (\Gamma+4)\xi^2 + (\Gamma+2)\xi] \\ -2\xi^3 + 3\xi^2 + \Gamma\xi \\ (l_e/2)[2\xi^3 + (\Gamma-2)\xi^2 - \Gamma\xi] \end{bmatrix}^T, \tag{32}$$

$$\overline{\mathbf{N}}_\mathrm{i}(\xi) = \left(\frac{1}{\Gamma+1}\right) \begin{bmatrix} (6/l_e)(\xi^2 - \xi) \\ 3\xi^2 - (\Gamma+4)\xi + \Gamma + 1 \\ (6/l_e)(\xi - \xi^2) \\ 3\xi^2 + (\Gamma-2)\xi \end{bmatrix}^T, \tag{33}$$

in which $\Gamma = 12EI/K_s G A l_e^2$.

Now, considering both ends of a nonlocal beam element connected to a linear spring and a torsional spring, the variational statement in Eq. (28) can be rewritten in terms of displacement expression in Eq. (29) for an element length $l_e$ of a uniform Timoshenko nonlocal beam:

$$\begin{aligned} 0 = &\Bigg\{ \int_0^1 \left[ \frac{EI}{l_e}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)'] + \frac{K_s G A}{l_e}[\mathbf{N}(\xi)' - l_e\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)' - l_e\overline{\mathbf{N}}(\xi)] \right] \mathrm{d}\xi \\ &+ (k_m)_i[\mathbf{N}(0)]^T[\mathbf{N}(0)] + (k_r)_i[\overline{\mathbf{N}}(0)]^T[\overline{\mathbf{N}}(0)] + (k_m)_j[\mathbf{N}(1)]^T[\mathbf{N}(1)] + (k_r)_j[\overline{\mathbf{N}}(1)]^T[\overline{\mathbf{N}}(1)] \Bigg\} \{\, \mathbf{v}\, \}_e \\ &+ \Bigg\{ \int_0^1 \Bigg[ I_0 l_e[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)] + I_2 l_e[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)] + \frac{\mu I_0}{l_e}\Big( [\mathbf{N}(\xi)' - l_e\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)'] \\ &- l_e[\overline{\mathbf{N}}(\xi)']^T[\mathbf{N}(\xi)] \Big) + \frac{\mu I_2}{l_e}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)'] \Bigg] \mathrm{d}\xi \Bigg\} \{\, \ddot{\mathbf{v}}\, \}_e \\ &= [\boldsymbol{k}]_e \{\, \mathbf{v}\, \}_e + [\boldsymbol{m}]_e \{\, \ddot{\mathbf{v}}\, \}_e \end{aligned} \tag{34}$$

where $[\mathbf{N}(\xi)'] = [\partial \mathbf{N}(\xi)/\partial \xi]$, subscripts $i$ and $j$ represent quantities specified at nodes $i$ and $j$, respectively. $[\boldsymbol{k}]_e$ and $[\boldsymbol{m}]_e$ are the element stiffness and mass matrices, respectively. In case of an Euler-Bernoulli nonlocal beam element, $\overline{\mathbf{N}}(\xi) = (1/l_e)\mathbf{N}(\xi)'$ and shear stress and rotatory inertia terms are neglected.

## 6 Numerical results

This section presents numerical examples for both Euler-Bernoulli and Timoshenko nonlocal elastically supported beams subjected to four classical boundary conditions (BCs): hinged-hinged, clamped-hinged, clamped-free and clamped-clamped. In all calculations were considered uniform rectangular cross-sectional beams of length $L = 10$ m, shear correction factor of $K_s = 5/6$, Poisson's ratio of $\nu = 0.3$. The beams are elastically restrained against rotational and translational displacement at both ends as shown in the Fig. 2. Non-dimensional stiffness parameters were defined as $\alpha = k_m(L^3/EI)$, $\beta = k_r(L^2/EI)$ and non-dimensional natural frequency as $\overline{\omega} = \omega \times L^2\sqrt{I_0/EI}$.

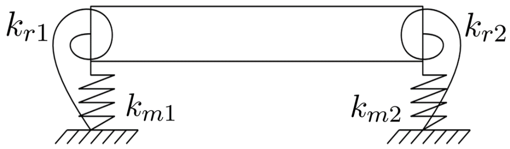


Figure 2. Elastically restrained beam

In order to investigate the influence of nonlocal parameter on the free-vibration characteristics of nonlocal beams, the first three non-dimensional frequencies are calculated for three different slenderness ratios by FEM. A convergence study indicated that results get converged with approximately 30 elements, so it was employed discretization of 40 elements in all computations. It was observed that infinite support stiffness can be obtained by setting the translational and rotational stiffness parameters equal to $1 \times 10^8$. Table 2 presents the boundary conditions expressed in terms of stiffness parameters $\alpha$ and $\beta$.

Table 2. Boundary conditions in terms of stiffness parameters $\alpha$ and $\beta$

| Boundary Condition | $\alpha_1$ | $\beta_1$ | $\alpha_2$ | $\beta_2$ |
|---|---|---|---|---|
| Hinged-Hinged | $1 \times 10^8$ | 0 | $1 \times 10^8$ | 0 |
| Clamped-Hinged | $1 \times 10^8$ | $1 \times 10^8$ | $1 \times 10^8$ | 0 |
| Clamped-Free | $1 \times 10^8$ | $1 \times 10^8$ | 0 | 0 |
| Clamped-Clamped | $1 \times 10^8$ | $1 \times 10^8$ | $1 \times 10^8$ | $1 \times 10^8$ |

Results for the effect of nonlocal parameter and slenderness ratio on the fundamental frequency of a hinged-hinged nonlocal beam are tabulated in Table 3. These are compared with the analytical solutions obtained by Reddy [4] and good agreement is noticed. Table 4 presents the results for the other three boundary conditions. It is observed that, as the nonlocal parameter $\mu$ increases, the fundamental frequency is decreased in all considered BCs, except for clamped-free case, in which it slightly increased. This behaviour agreed with the results reported by Eltaher et al. [9]. Observe that Timoshenko beam presents lower frequency values than Euler-Bernoulli beam and the difference between them becomes more expressive as the $L/h$ ratio decreases. This results are presented in graphical form in Fig. 3.

Table 3. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on first non-dimensional frequency ($\overline{\omega}_1$) of uniform hinged-hinged beam

| $L/h$ | $\mu$ (m$^2$) | EBT | EBT$^{(1)}$ | TBT | TBT$^{(1)}$ |
|---|---|---|---|---|---|
| 100 | 0 | 9.8696 | 9.8696 | 9.8679 | 9.8683 |
| | 1 | 9.4159 | 9.4159 | 9.4143 | 9.4147 |
| | 2 | 9.0197 | 9.0195 | 9.0180 | 9.0183 |
| 20 | 0 | 9.8696 | 9.8696 | 9.8281 | 9.8381 |
| | 1 | 9.4159 | 9.4159 | 9.3763 | 9.3858 |
| | 2 | 9.0195 | 9.0195 | 8.9816 | 8.9907 |
| 10 | 0 | 9.8696 | 9.8696 | 9.7075 | 9.7454 |
| | 1 | 9.4159 | 9.4159 | 9.2613 | 9.2973 |
| | 2 | 9.0195 | 9.0195 | 8.8714 | 8.9059 |

$^{(1)}$ Reddy [4].

Table 4. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on first non-dimensional frequency ($\overline{\omega}_1$) of uniform beam for various BCs

| BC | Clamped-Hinged | | Clamped-Free | | Clamped-Clamped | |
|---|---|---|---|---|---|---|
| $\mu$ (m$^2$) | EBT | TBT | EBT | TBT | EBT | TBT |
| $L/h = 100$ | | | | | | |
| 0 | 15.4182 | 15.4120 | 3.5160 | 3.5157 | 22.3733 | 22.3577 |
| 1 | 14.5998 | 14.5934 | 3.5299 | 3.5310 | 21.1090 | 21.0946 |
| 2 | 13.8944 | 13.8907 | 3.5458 | 3.5466 | 20.0328 | 20.0193 |
| $L/h = 20$ | | | | | | |
| 0 | 15.4182 | 15.2658 | 3.5160 | 3.5090 | 22.3733 | 21.9954 |
| 1 | 14.5990 | 14.4569 | 3.5304 | 3.5241 | 21.1090 | 20.7584 |
| 2 | 13.8983 | 13.7622 | 3.5464 | 3.5395 | 20.0328 | 19.7038 |
| $L/h = 10$ | | | | | | |
| 0 | 15.4182 | 14.8363 | 3.5160 | 3.4884 | 22.3733 | 20.9729 |
| 1 | 14.5990 | 14.0557 | 3.5304 | 3.5027 | 21.1090 | 19.8086 |
| 2 | 13.8983 | 13.3843 | 3.5464 | 3.5174 | 20.0328 | 18.8127 |

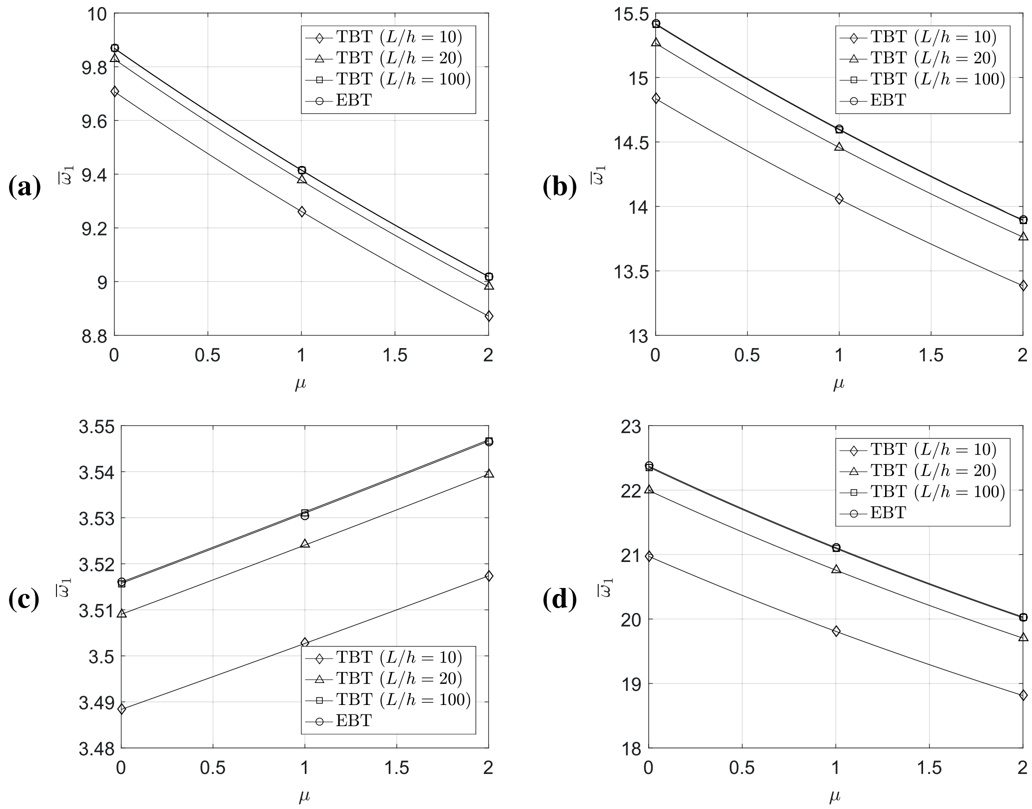


Figure 3. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on first non-dimensional frequency ($\overline{\omega}_1$) of uniform beam for **(a)** hinged-hinged, **(b)** clamped-hinged, **(c)** clamped-free, **(d)** clamped-clamped.

Table 5 and Table 6 presents, respectively, the influence of $\mu$ and $L/h$ ratio on the second and third non-dimensional natural frequencies of EBT and TBT. In all considered BCs, this frequencies decreased as the nonlocal parameter increased. Notice that the reduction effect becomes more pronounced at higher frequencies than in the first one. The clamped-clamped case presents the largest relative decrease in the natural frequencies within the same increase of nonlocal parameter. This results are plotted in Fig. 4 and Fig. 5.

Table 5. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on second non-dimensional frequency ($\overline{\omega}_2$) of uniform beam for various BCs

| BC | Hinged-Hinged | | Clamped-Hinged | | Clamped-Free | | Clamped-Clamped | |
|---|---|---|---|---|---|---|---|---|
| $\mu$ (m$^2$) | EBT | TBT | EBT | TBT | EBT | TBT | EBT | TBT |
| | | | | $L/h = 100$ | | | | |
| 0 | 39.4784 | 39.4517 | 49.9648 | 49.9098 | 22.0345 | 22.0222 | 61.6727 | 61.5740 |
| 1 | 33.4277 | 33.4051 | 41.7941 | 41.7500 | 20.6803 | 20.6679 | 50.9832 | 50.9047 |
| 2 | 29.5110 | 29.4912 | 36.6583 | 36.6179 | 19.5099 | 19.4985 | 44.3947 | 44.3276 |
| | | | | $L/h = 20$ | | | | |
| 0 | 39.4784 | 38.8309 | 49.9648 | 48.6494 | 22.0345 | 21.7347 | 61.6727 | 59.3476 |
| 1 | 33.4277 | 32.8794 | 41.7950 | 40.7250 | 20.6799 | 20.3935 | 50.9832 | 49.1356 |
| 2 | 29.5111 | 29.0271 | 36.6545 | 35.7301 | 19.5098 | 19.2366 | 44.3947 | 42.8126 |
| | | | | $L/h = 10$ | | | | |
| 0 | 39.4784 | 37.0995 | 49.9648 | 45.3067 | 22.0345 | 20.9075 | 61.6727 | 53.7570 |
| 1 | 33.4277 | 31.4133 | 41.7950 | 38.0014 | 20.6799 | 19.6065 | 50.9832 | 44.6785 |
| 2 | 29.5111 | 27.7327 | 36.6545 | 33.3710 | 19.5098 | 18.4872 | 44.3947 | 38.9978 |

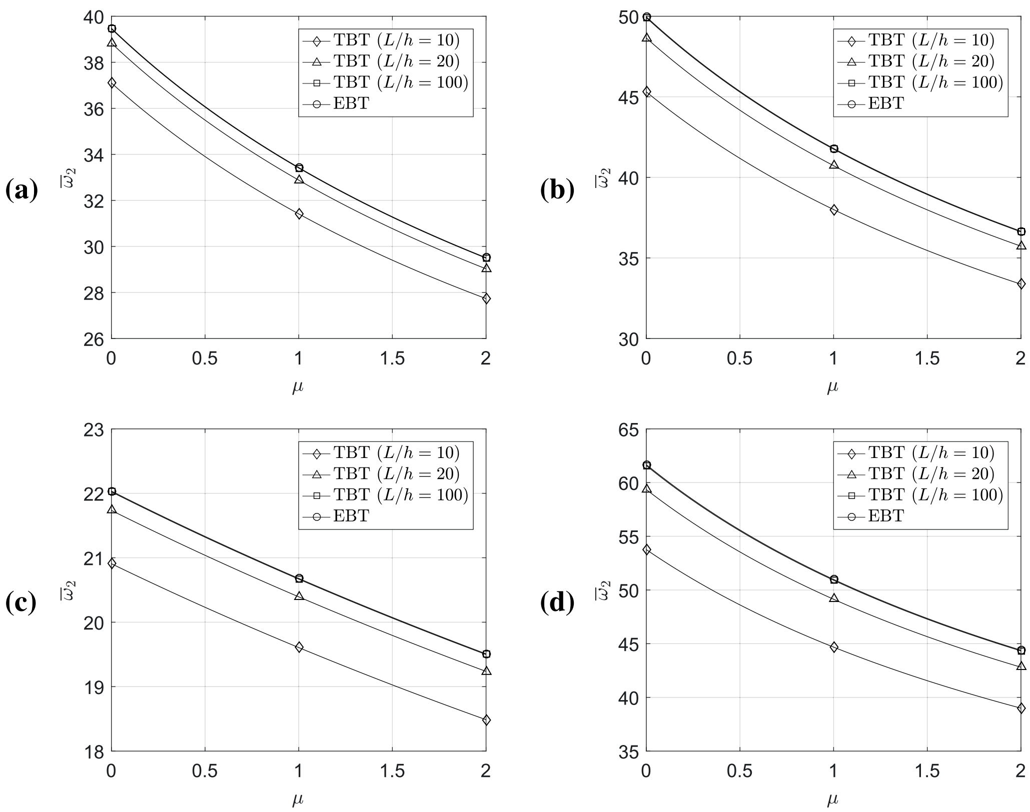


Figure 4. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on second non-dimensional frequency ($\overline{\omega}_2$) of uniform beam for **(a)** hinged-hinged, **(b)** clamped-hinged, **(c)** clamped-free, **(d)** clamped-clamped.

Table 6. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on third non-dimensional frequency ($\overline{\omega}_3$) of uniform beam for various BCs

| BC | Hinged-Hinged | | Clamped-Hinged | | Clamped-Free | | Clamped-Clamped | |
|---|---|---|---|---|---|---|---|---|
| $\mu$ (m$^2$) | EBT | TBT | EBT | TBT | EBT | TBT | EBT | TBT |
| $L/h = 100$ | | | | | | | | |
| 0 | 88.8265 | 88.6919 | 104.2477 | 104.0256 | 61.6972 | 61.6157 | 120.9032 | 120.5625 |
| 1 | 64.6415 | 64.5435 | 74.8522 | 74.6977 | 51.0635 | 50.9951 | 85.7167 | 85.4872 |
| 2 | 53.3079 | 53.2272 | 61.4735 | 61.3484 | 44.5604 | 44.4995 | 70.1226 | 69.9368 |
| $L/h = 20$ | | | | | | | | |
| 0 | 88.8265 | 85.6724 | 104.2477 | 99.1445 | 61.6972 | 59.7638 | 120.9032 | 113.2368 |
| 1 | 64.6415 | 62.3460 | 74.8517 | 71.3041 | 51.0638 | 49.4384 | 85.7166 | 80.5481 |
| 2 | 53.3079 | 51.4149 | 61.4747 | 58.5830 | 44.5602 | 43.1256 | 70.1225 | 65.9426 |
| $L/h = 10$ | | | | | | | | |
| 0 | 88.8265 | 78.1860 | 104.2477 | 87.7646 | 61.6972 | 54.9992 | 120.9032 | 97.2032 |
| 1 | 64.6415 | 56.8962 | 74.8517 | 63.3626 | 51.0638 | 45.4622 | 85.7166 | 69.6686 |
| 2 | 53.3079 | 46.9201 | 61.4747 | 52.1145 | 44.5602 | 39.6323 | 70.1225 | 57.1564 |

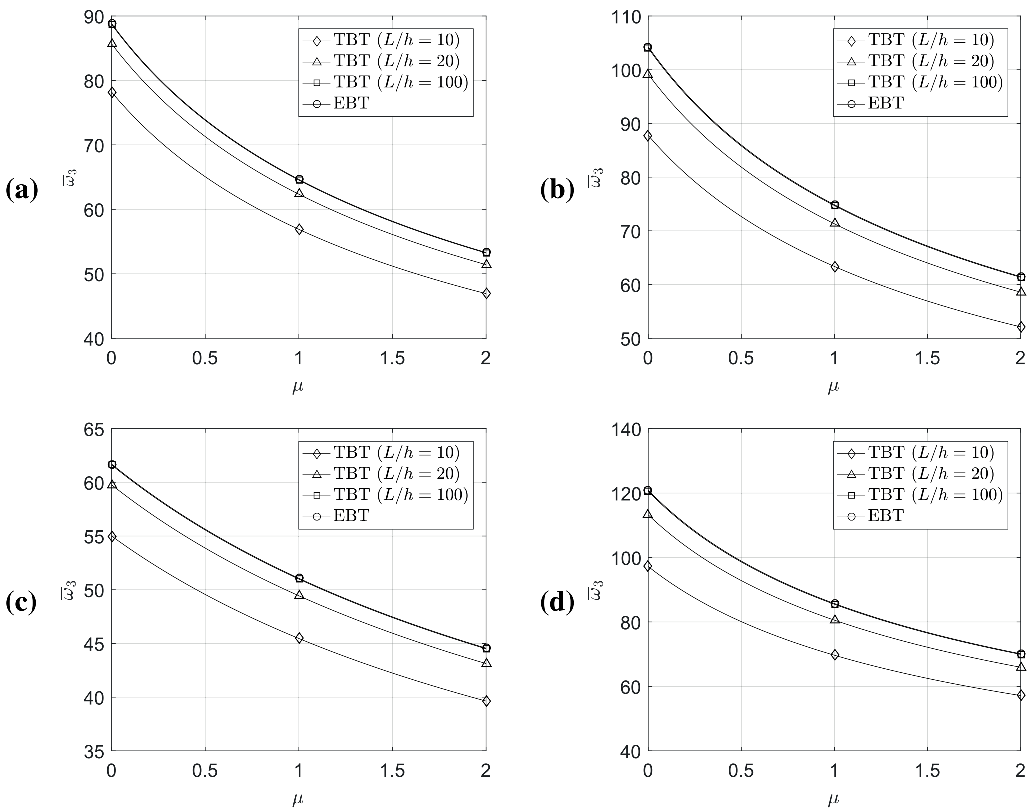


Figure 5. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on third non-dimensional frequency ($\overline{\omega}_3$) of uniform beam for **(a)** hinged-hinged, **(b)** clamped-hinged, **(c)** clamped-free, **(d)** clamped-clamped.

The effect of support stiffness on the free vibration characteristics of local and nonlocal ($\mu = 3$) beams is presented in Table 7. Results for the first non-dimensional frequency of Timoshenko beams are presented for different values of stiffness parameters. It was considered a slenderness ratio of $L/h = 20$, $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$.

Table 7. Influence of nonlocal parameter $\mu$ and $L/h$ ratio on non-dimensional fundamental frequency of hinged-hinged beam

| $\overline{\omega}_1$ | | $\beta$ | | | | |
|---|---|---|---|---|---|---|
| $\alpha$ | $\mu$ (m$^2$) | $10^0$ | $10^2$ | $10^4$ | $10^6$ | $10^8$ |
| $10^0$ | 0 | 1.4026 | 1.4078 | 1.4121 | 1.4122 | 1.4122 |
| | 3 | 1.3992 | 1.4063 | 1.4121 | 1.4122 | 1.4122 |
| $10^2$ | 0 | 8.2821 | 9.9348 | 12.2605 | 12.3170 | 12.3176 |
| | 3 | 7.5467 | 9.2642 | 12.0472 | 12.1232 | 12.1240 |
| $10^4$ | 0 | 9.8579 | 13.1699 | 21.4877 | 21.8088 | 21.8121 |
| | 3 | 8.6629 | 11.5661 | 18.4638 | 18.7062 | 18.7087 |
| $10^6$ | 0 | 9.8772 | 13.2148 | 21.6623 | 21.9902 | 21.9936 |
| | 3 | 8.6760 | 11.5950 | 18.5447 | 18.7885 | 18.7910 |
| $10^8$ | 0 | 9.8774 | 13.2152 | 21.6641 | 21.9920 | 21.9954 |
| | 3 | 8.6761 | 11.5953 | 18.5455 | 18.7893 | 18.7918 |

Results for local beam are in good agreement with the classical theory presented by Kocatürk and Şimşek [17]. Note that the nonlocal effect decrease as the support stiffness decreases and becomes negligible for small values of stiffness parameters, $\alpha$ and $\beta$.

## 7 Conclusions

In this paper, variational formulation for transverse free vibration of elastically supported Euler-Bernoulli and Timoshenko beams based on Eringen's nonlocal differential elasticity theory have been presented. A two-node nonlocal beam element with two degree of freedom per node was developed based upon Hamilton's principle. It was observed the possibility to simulate infinite support stiffness by setting the non-dimensional translational and rotational support stiffness parameters equal to $1 \times 10^8$. The influence of support stiffness, nonlocal parameter and slenderness ratio on the dynamic behaviour was investigated. Nonlocal effect decreases the natural frequencies of nanobeams for all classical boundary conditions, except for the clamped-clamped case, where a increment on the fundamental frequency was observed. Also, it was observed that the nonlocal effect becomes negligible for low values of support stiffness parameters. Numerical results obtained for local and nonlocal beam are in good agreement with other researchers results.

## 8 Permission

The authors are the only responsible for the printed material included in this paper.

## References


[1] Eringen, A. C. & Edelen, D., 1972. On nonlocal elasticity. *International Journal of Engineering Science*, vol. 10, n. 3, pp. 233–248.

[2] Eringen, A. C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of applied physics*, vol. 54, n. 9, pp. 4703–4710.

[3] Xu, M., 2006. Free transverse vibrations of nano-to-micron scale beams. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 462, n. 2074, pp. 2977–2995.

[4] Reddy, J., 2007. Nonlocal theories for bending, buckling and vibration of beams. *International Journal of Engineering Science*, vol. 45, n. 2-8, pp. 288–307.

[5] Civalek, Ö., Demir, Ç., & Akgöz, B., 2010. Free vibration and bending analyses of cantilever microtubules based on nonlocal continuum model. *Mathematical and Computational Applications*, vol. 15, n. 2, pp. 289–298.

[6] Ehteshami, H. & Hajabasi, M. A., 2011. Analytical approaches for vibration analysis of multi-walled carbon nanotubes modeled as multiple nonlocal euler beams. *Physica E: Low-dimensional Systems and Nanostructures*, vol. 44, n. 1, pp. 270–285.

[7] Pradhan, S., 2012. Nonlocal finite element analysis and small scale effects of cnts with timoshenko beam theory. *Finite Elements in Analysis and Design*, vol. 50, pp. 8–20.

[8] Aydogdu, M., 2012. Longitudinal wave propagation in nanorods using a general nonlocal unimodal rod theory and calibration of nonlocal parameter with lattice dynamics. *International Journal of Engineering Science*, vol. 56, pp. 17–28.

[9] Eltaher, M., Alshorbagy, A. E., & Mahmoud, F., 2013. Vibration analysis of euler–bernoulli nanobeams by using finite element method. *Applied Mathematical Modelling*, vol. 37, n. 7, pp. 4787–4797.

[10] Nazemnezhad, R. & Hosseini-Hashemi, S., 2014. Nonlocal nonlinear free vibration of functionally graded nanobeams. *Composite Structures*, vol. 110, pp. 192–199.

[11] Boumia, L., Zidour, M., Benzair, A., & Tounsi, A., 2014. A timoshenko beam model for vibration analysis of chiral single-walled carbon nanotubes. *Physica E: Low-dimensional Systems and Nanostructures*, vol. 59, pp. 186–191.

[12] Rahmanian, M., Torkaman-Asadi, M., Firouz-Abadi, R., & Kouchakzadeh, M., 2016. Free vibrations analysis of carbon nanotubes resting on winkler foundations based on nonlocal models. *Physica B: Condensed Matter*, vol. 484, pp. 83–94.

[13] Demir, Ç. & Civalek, Ö., 2017. A new nonlocal fem via hermitian cubic shape functions for thermal vibration of nano beams surrounded by an elastic matrix. *Composite Structures*, vol. 168, pp. 872–884.

[14] Chun, K., 1972. Free vibration of a beam with one end spring-hinged and the other free. *Journal of Applied Mechanics*, vol. 39, pp. 1154.

[15] Abbas, B., 1984. Vibrations of timoshenko beams with elastically restrained ends. *Journal of Sound and Vibration*, vol. 97, n. 4, pp. 541–548.

[16] Craver Jr., W. & Jampala, P., 1993. Transverse vibrations of a linearly tapered cantilever beam with constraining springs. *Journal of Sound and Vibration*, vol. 166, n. 3, pp. 521–529.

[17] Kocatürk, T. & Şimşek, M., 2005. Free vibration analysis of timoshenko beams under various boundary conditions. *Sigma*, vol. 1, pp. 30–44.

[18] Azevêdo, A., Vasconcelos, A., & dos Santos, S., 2016. Dynamic analysis of elastically supported timoshenko beam. *Revista Interdisciplinar de Pesquisa em Engenharia*, vol. 2 (34), pp. 71–85.

[19] Kiani, K., 2016. Nonlocal-integro-differential modeling of vibration of elastically supported nanorods. *Physica E: Low-dimensional Systems and Nanostructures*, vol. 83, pp. 151–163.

[20] Timoshenko, S. P., 1921. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philosophical Magazine*, vol. 41, pp. 744–746.