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Conference Paper · January 2017

DOI: 10.26678/ABCM.COBEM2017.COB17-1084

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COBEM2017-1084
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Abstract. *Free vibration analysis of beams on the elastic foundation is necessary for an optimal design in engineering applications, especially in the fields of transportation systems. This paper analyzes the influence of the foundation parameters and the boundary conditions configuration in the dynamic response of Euler-Bernoulli beam on Pasternak foundation. A finite element solution is developed and compared with the analytic solution to verify the reliability of the method in this kind of problem. The results showed that Pasternak foundation increases the frequency parameters of the beam. Also, the presence of the foundation reduces the normal mode amplitude and the clamped-free case is the most affected by the presence of a Pasternak foundation.*

Keywords: *Euler-Bernoulli Beam Theory, Finite Element Method, Elastic Foundation, Free Vibration Analysis, Pasternak Foundation Model.*

1. INTRODUCTION

The response of the soil-structure system represents one of the major problems encountered in structural dynamic due to its intrinsic mathematical complexity. The search for a physically consistent model that gives a reliable information for the soil-structure interaction has been the greatest challenge of the researches through the years.

To comprehend the nature of the soil-structure interrelationship is fundamental to consider the mechanical behavior of beam, the behavior of the soil subgrade as well as the interaction between them. In addition, the inability in determining all input parameters of soil media is a determinant factor in the understanding of its behavior, since it presents heterogeneous, anisotropic and nonlinear characteristics. In this context, the mechanical model, which consider the soil-structure interaction as a beam on elastic foundation, emerge as an alternative widely used in the design of structures.

The main scope of mechanical approach bases in the Winkler model, which represent the soil media as a system of closely spaced, discrete, linearly elastic springs and the force linearly proportional to the beam displacement (Selvadurai, 1979). However, Winkler model is a crude approximation of the true behavior as it is not capable of modelling the cohesion of the soil. To overcome this limitation, many idealized models, which include the effect of continuity and cohesion of the soil, were developed (Filonenko-Borodich, 1940; Hetenyi, 1946; Pasternak, 1954; Kerr, 1965). The Pasternak model gives a more accurate response of the soil media since it considers the existence of shear interaction among the springs (Dutta and Roy, 2002). This model is the most used as it can simulate a large variety of types of soil, representing a generalized foundation (Kerr, 1964).

Concerning the analysis of the mechanical behavior of the beam, the so-called classical Euler-Bernoulli is adequate for slender beams at lower modes of vibrations. In this theory, straight and normal lines remain the straight and normal after deformation. Although disregarding the effect of shear deformation and the rotatory inertia factors, predicted in Timoshenko beam theory, the Euler-Bernoulli beam theory presents good accuracy for most of the engineering applications (Fernandes *et al.*, 2016).

Several papers concerned investigations this field. Zhaohua and Cook (1983) developed two kinds of finite elements to analyze a Euler-Bernoulli beam on one- and two-parameter foundation and verified the accuracy of them to the exact solution. De Rosa and Maurizi (1998) developed the analytic solution for the free vibration frequencies of a Euler-Bernoulli beam on Pasternak foundation in the presence of flexible ends and of a concentrated mass at an arbitrary position at its span. Ho and Chen (1998) studied the free and forced response of non-uniform Euler-Bernoulli beams resting on non-homogeneous Winkler foundation. Morfidis and Avramidis (2002) developed a finite element for the analysis of reinforced concrete and steel structures. Kumar and Reddy (2016) analyzed the free torsional vibrations of doubly symmetric thin-walled beams of open section and resting on Winkler and Pasternak foundations.

The aim of this paper is to analyze the effect of the Pasternak foundation in the dynamic response of an Euler-Bernoulli beam. A finite element is developed and compared with the analytic solution. Several numerical examples presents the influence of the Pasternak foundation on the free vibration of a Euler-Bernoulli beam for various boundary conditions, and foundation elastic and shear layer stiffness.

2. CLASSIC THEORY

Figure 1 illustrates the system of a uniform Euler-Bernoulli beam resting on a Pasternak foundation. The potential energy of the beam-foundation system is given by (Soares and Hoefel, 2015):

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2\nu(x,t)}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L k_f\left(\nu\left(x,t\right)\right)^2 dx + \frac{1}{2}\int_0^L G_p\left(\frac{\partial\nu\left(x,t\right)}{\partial x}\right)^2 dx, \tag{1}$$

where L is the length of beam, I, the moment of inertia of cross section, E, the modulus of elasticity, k_f the foundation elastic stiffness coefficient, G_p the shear layer stiffness, and $\nu(x,t)$ is the transverse deflection at the axial location x and time t.

Considering the inertia of translation, the kinetic energy is given by:

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial\nu(x,t)}{\partial t}\right)^2 dx, \tag{2}$$

where ρ is the mass per unit volume and A is the cross-sectional area. The equation of motion can be obtained using Hamilton's principle:

$$\int_{t_1}^{t_2}\delta(T-U)dt + \int_{t_1}^{t_2}\delta W_{nc}dt = 0, \tag{3}$$

where δW_{nc} is the virtual work due non conservative forces, t_1 and t_2 are times at which the configuration of the system is known and δ is the symbol denoting virtual change. Substituting Eqs. (1) and (2) on the Eq. (3), after some manipulations, one can obtain the differential equation for free vibration response:

$$EI\frac{\partial^4\nu\left(x,t\right)}{\partial x^4} + \rho A\frac{\partial^2\nu\left(x,t\right)}{\partial t^2} + k_f\nu\left(x,t\right) - G_p\frac{\partial^2\nu\left(x,t\right)}{\partial x^2} = 0. \tag{4}$$

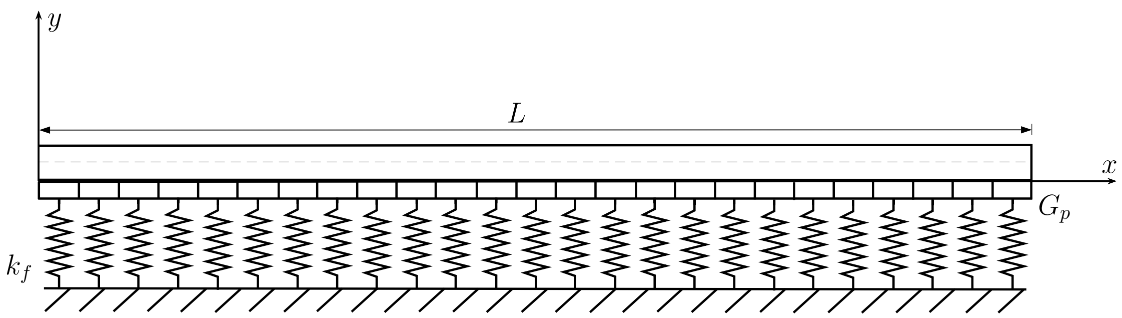


Figure 1. A beam on a Pasternak foundation.

Assuming that the beam is excited harmonically with an angular frequency ω and:

$$\nu(x,t) = V(x)\cdot e^{i\omega t}, \quad \xi = x/L, \text{ and } b^2 = \frac{\rho A L^4}{EI}\omega^2, \tag{5}$$

where $i = \sqrt{-1}$, ξ is the non-dimensional length of the beam, $V(x)$ is the normal function of $\nu(x,t)$ and b is the frequency parameter. Substituting the relations presented in Eqs. (5) into Eq. (4) and omitting the common term $e^{i\omega t}$ we obtain (De Rosa and Maurizi, 1998):

$$\frac{d^4V(\xi)}{d\xi^4} - p^2\frac{d^2V(\xi)}{d\xi^2} + (e^2-b^2)V(\xi) = 0, \tag{6}$$

where e and p are the coefficients related with the elastic and the shear layer stiffness, respectively, given '1234590by:

$$e^2 = \frac{k_f L^4}{EI} \text{ and } p^2 = \frac{G_p L^2}{EI}. \tag{7}$$

The solution of the O.D.E. of Eq. (6) leads to the displacement being expressed in trigonometric and hyperbolic functions:

$$V(\xi) = C_1\, cosh(\alpha\xi) + C_2\, sinh(\alpha\xi) + C_3\, cos(\beta\xi) + C_4\, sin(\beta\xi), \tag{8}$$

where:

$$\alpha = \frac{\sqrt{2}}{2}\sqrt{\ p^2 + \sqrt{p^4 - 4(e^2 - b^2)}}, \tag{9}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{-p^2 + \sqrt{p^4 - 4(e^2 - b^2)}}, \tag{10}$$

and C is a constant.

Equations 6 shows that the beam-foundation theory represents a generalization of the beam theory. Disregarding the parameters e and p, the solution regress to the solution of a beam without foundation. Also, the Pasternak foundation theory is a higher generalization as it includes the solution for Winkler when $p = 0$.

3. FINITE ELEMENT FORMULATION

Consider a uniform Euler-Bernoulli beam element on Pasternak Foundation as shown in Fig. 2. The beam element consists of two nodes and each node has two degrees of freedom: V, the total deflection, and Ψ, the slope due to bending.

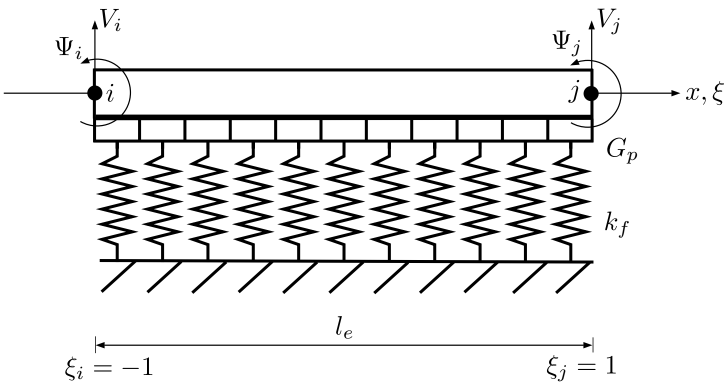


Figure 2. Beam on Pasternak foundation element

Solving the homogeneous form of Euler-Bernoulli beam static equations, one can obtain a cubic displacement functions as follows (Zhaohua and Cook, 1983):

$$V_i(\xi) = \sum_{i=0}^{3} \lambda_i \xi^i, \tag{11}$$

where λ_i is a constant.

Using the non-dimension coordinate, ξ, and element length, l_e, the matrix form of the displacement can be written as:

$$V = [\,\mathbf{N}(\xi)\,]\{\,\mathbf{v}\,\}_e, \tag{12}$$

where $[\,\mathbf{N}(\xi)\,]$ is the shape functions and $\{\,\mathbf{v}\,\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for a single element.

Therefore, the shape functions in Eq. (12) can be expressed as:

$$\mathbf{N_i}(\xi) = \frac{1}{4}\left\{\begin{array}{c} 2 - 3\xi + \xi^3 \\ (l_e/2)\,[1 - \xi - \xi^2 + \xi^3] \\ 2 + 3\xi - \xi^3 \\ (l_e/2)\,[-1 - \xi + \xi^2 + \xi^3] \end{array}\right\}. \tag{13}$$

Thus, considering the foundation and the beam, the potential and kinetic energy for an element length l_e are given by:

$$\boldsymbol{U_e} = \frac{1}{2}\frac{2EI}{l_e}\int_{-1}^{1}\left(\frac{\partial^2 V}{\partial \xi^2}\right)^2 d\xi + \frac{1}{2}\frac{k_f l_e}{2}\int_{-1}^{1} V^2 d\xi + \frac{1}{2}\frac{2G_p}{l_e}\int_{-1}^{1}\left(\frac{\partial V}{\partial \xi}\right)^2 d\xi \text{ and} \tag{14}$$

$$\boldsymbol{T_e} = \frac{1}{2}\frac{\rho A l_e}{2}\int_{-1}^{1}\left(\frac{\partial V}{\partial t}\right)^2 d\xi. \tag{15}$$

Substituting the displacement expression, Eq. (12), into the potential energy, Eq. (14), gives:

$$\begin{aligned}\boldsymbol{U_e} &= \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2EI}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)'']^T[\mathbf{N}(\xi)'']d\xi\right]\{\mathbf{v}\}_e + \\ &\quad \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{k_f l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\,d\xi + \frac{2G_p}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']\,d\xi\right]\{\mathbf{v}\}_e,\end{aligned} \tag{16}$$

where $[\mathbf{N}(\xi)'] = [\partial\mathbf{N}(\xi)/\partial\xi]$. Therefore, the element stiffness matrix is given by:

$$[\boldsymbol{k_e}] = \left[\frac{2EI}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)'']^T[\mathbf{N}(\xi)'']d\xi\right. + \tag{17}$$

$$\left.\frac{k_f l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\,d\xi + \frac{2G_p}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']\,d\xi\right]. \tag{18}$$

Substituting the displacement expression, Eq. (12), into the kinetic energy, Eq. (15), gives:

$$\boldsymbol{T_e} = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\frac{\rho A l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e, \tag{19}$$

hence, the element mass matrix is given by:

$$[\boldsymbol{m_e}] = \left[\frac{\rho A l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi\right]. \tag{20}$$

4. NUMERICAL RESULTS

To show the effects of the foundation parameters on the natural frequencies of a beam on elastic foundation, some numerical examples are presented. The same parameter values are used for all examples, except when specified. Consider a beam of uniform cross-section, such that $E = 210\,GPa$, $\rho = 7850\,kg/m^3$, $h/L = 0.05\,m$ and $L = 2\,m$.

Table 1. Comparison table for the frequency parameters of analytic and FEM analyses.

Mode Number	Without Foundation	Winkler			Pasternak		
		Analytic	FEM-30e	FEM-70e	Analytic	FEM-30e	FEM-70e
1	9.870	11.064	11.064	11.064	19.213	19.213	19.213
2	39.478	39.794	39.794	39.794	50.700	50.700	50.700
3	88.826	88.967	88.968	88.967	100.677	100.677	100.677
4	157.914	157.993	157.996	157.993	170.028	170.031	170.028
5	246.740	246.791	246.804	246.791	258.987	258.999	258.987

To analyze the influence a foundation in the frequency parameters, Table 1 presents the comparison between analytic and FEM solutions for a hinged-hinged beam. The second column presents the frequency parameters for a beam without foundation ($e = 0$, $p = 0$), third to sixth, for Winkler foundation ($e = 5$, $p = 0$), and seventh to tenth, for Pasternak foundation ($e = 5$, $p = 5$).

The results showed that the foundation increases the frequency parameter and Pasternak presents the highest increase. Also, the Winkler foundation increase reduces drastically as the mode number rise. This reduction can also be observed in the frequency parameters of Pasternak foundation. However, the decrease is not as high as presented by Winkler.

For higher modes, the difference between the FEM and analytic solutions decreases when the number of elements is increased and with 70 elements the FEM and analytic solutions present a excellent agreement. Therefore, FEM formulation presents a high accuracy in this kind of problem.

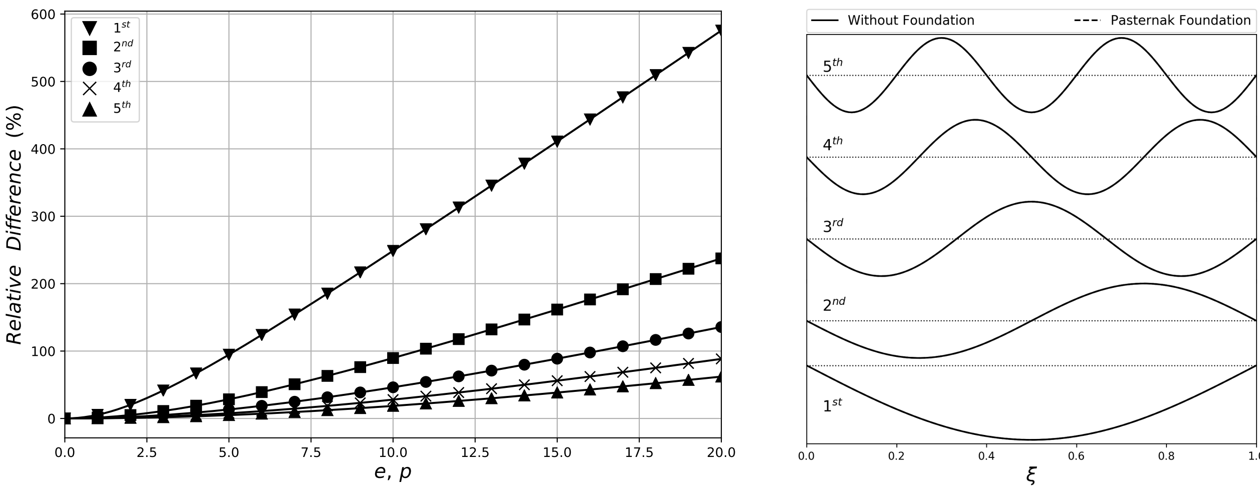
The subsections present numerical examples for hinged-hinged, clamped-clamped and clamped-free beams for the FEM solution using 70 elements to analyze the effect of the Pasternak foundation in the usual boundary conditions of the Euler-Bernoulli Beam. The boundary conditions are defined in Table 2.

Table 2. Boundary Conditions

Boundary Condition	Deflection	Slope	Moment	Shear Force
Hinged	$V(\xi) = 0$	-	$\frac{d^2V(\xi)}{d\xi^2} = 0$	-
Clamped	$V(\xi) = 0$	$\frac{dV(\xi)}{d\xi} = 0$	-	-
Free	-	-	$\frac{d^2V(\xi)}{d\xi^2} = 0$	$EI\,\frac{d^3V(\xi)}{d\xi^3} + G_p\,\frac{dV(\xi)}{d\xi} = 0$

4.1 Hinged-Hinged

Figure 3(a) shows the relative difference between the frequency parameters of a hinged-hinged on a Pasternak foundation with the frequency parameters of a beam without foundation ($e = 0$, $p = 0$) for various values of the parameter e and p for the first five frequencies. This relative difference illustrates the increase in the natural frequencies for a beam on Pasternak foundation with increasing stiffness.



(a) Increase in the natural frequencies.

(b) First five normal modes.

Figure 3. Pasternak foundation influence on a hinged-hinged beam.

As the foundation stiffness increases, the natural frequencies have a huge increase. Also, the reduction as the number mode increases, presented in Table 1, can be noted and makes possible distinguish that the first mode has the major increases, in which the difference between its frequency parameters and the others becomes more significant as the foundation stiffness rise.

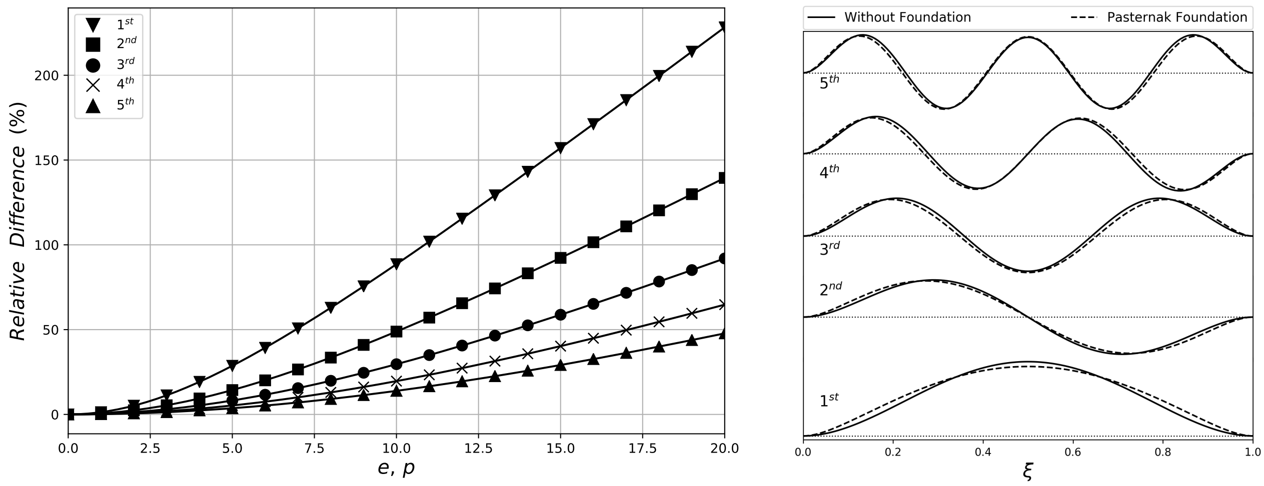
Figure 3(b) illustrates the normal modes for the first five frequencies for a beam without foundation and a beam on a Pasternak foundation ($e = 20$, $p = 20$). The figure shows that the Pasternak foundation does not affect the normal modes, even though the foundation parameters used represent a very stiff foundation.

4.2 Clamped-Clamped

The relative difference between the frequency parameters for the first five frequencies of a clamped-clamped beam on Pasternak foundation with a beam without foundation as the foundation stiffness increases is shown in Fig. 4(a).

Even though the frequencies still have a huge increase, it is not high as the presented by the hinged-hinged case. Also, the normal modes presents a slight difference when the foundation is present, as shown in Fig. 4(b). In contrast to the hinged-hinged case, all the modes present a reduction in the amplitude with the first normal mode presenting the the major decrease.

Figure 5 present the relative error between the amplitude of a clamped-clamped beam on a Pasternak foundation and a beam without foundation as the foundation stiffness increases. The figure shows that higher normal modes appears to converges to same curve. This behaviour allows inferring that as the mode number rise, the normal modes will presents the same reduction.



(a) Increase in the natural frequencies. (b) First five normal modes.

Figure 4. Pasternak foundation influence on a hinged-hinged beam.

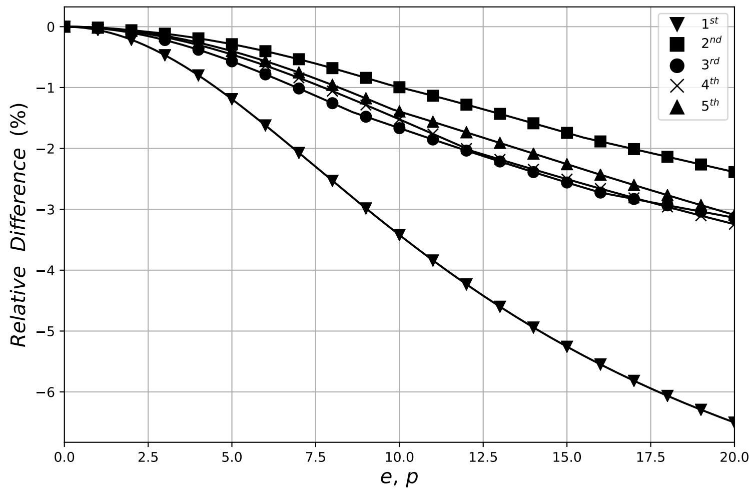


Figure 5. Relative difference between the maximum amplitude of a beam on a Pasternak foundation and a beam without foundation for the clamped-clamped boundary condition.

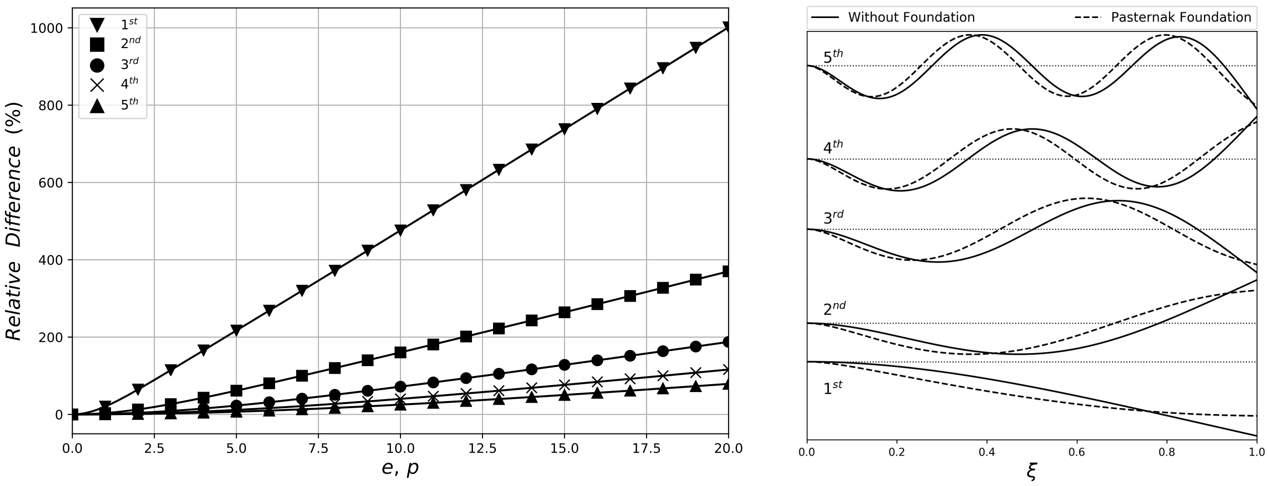
4.3 Clamped-Free

In contrast to observed in the hinged-hinged and clamped-clamped, the clamped-free beam presents a major increase, even higher than that from hinged-hinged. As noted in the other boundary conditions, the first mode presents the great increase when compared to the others, ratifying this characteristics of the foundation to affect only the lower modes with the range of influence increasing as the foundation stiffness increases.

Figures 6(b) and 7 shows the effect of the Pasternak foundation in the normal modes of a clamped-free beam. This boundary conditions presents the major reduction in the amplitude. In contrast to the expected, for some foundation stiffness, the amplitude of the normal mode increase with the presence of the Pasternak foundation. However, only the first mode does present this behaviour, reducing the amplitude for all the foundation frequencies values. This behavior can be explained by the extra factor that the Pasternak foundation insert in the free-end boundary condition, as seen in Table 2.

5. CONCLUSIONS

This paper presented a finite element method for free vibration analysis of Euler-Bernoulli beams on Pasternak foundation regarding the foundation parameters and the boundary conditions. A cubic polynomial was used for deflection in a two-node beam element with two degrees of freedom per node. The investigation determined that the presence of a foundation increase the natural frequencies of the beam vibration. The Pasternak foundation presents higher frequencies than the Winkler foundation, and the increase is expressive for all modes. Also, the presence of the foundation also have



(a) Increase in the natural frequencies.

(b) First five normal modes.

Figure 6. Pasternak foundation influence on a hinged-hinged beam.

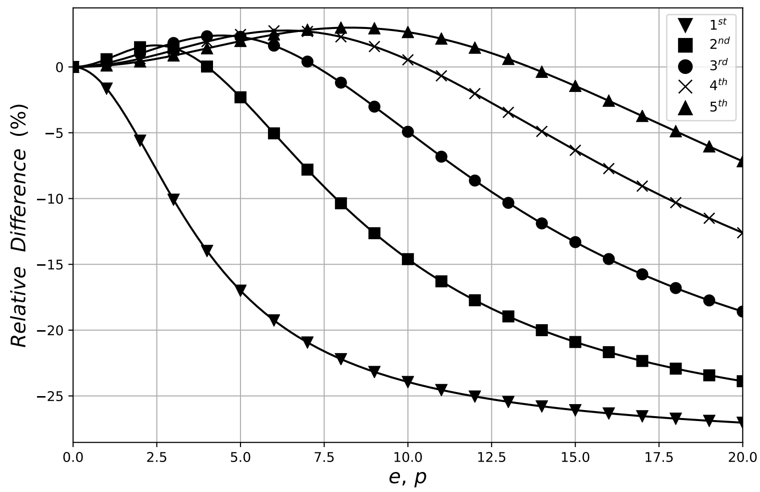


Figure 7. Relative difference between the maximum amplitude of a beam on a Pasternak foundation and a beam without foundation for the clamped-free boundary condition.

influence in the normal modes, reducing the amplitude as the foundation stiffness increases. The clamped-free case if the most affected by the presence of a Pasternak foundation, which present a increase in the amplitude for some foundation stiffness.

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